Lyapunov criteria for uniform convergence of conditional distributions of absorbed Markov processes

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April 6, 2017

Abstract

We study the quasi-stationary behavior of multidimensional processes absorbed when one of the coordinates vanishes. Our results cover competitive or weakly cooperative Lotka-Volterra birth and death processes and Feller diffusions with competitive Lotka-Volterra interaction. To this aim, we develop original non-linear Lyapunov criteria involving two Lyapunov functions, which apply to general Markov processes.

Keywords: stochastic Lotka-Volterra systems; multitype population dynamics; multidimensional birth and death process; multidimensional Feller diffusions; process absorbed on the boundary; quasi-stationary distribution; uniform exponential mixing property; Lyapunov function

2010 Mathematics Subject Classification. Primary: 60J27; 37A25; 60B10. Secondary: 92D25; 92D40.

1 Introduction

We consider a Markov process $(X_t, t \geq 0)$ on a topological state space $E \cup \partial$, where $\partial \cap E = \emptyset$ and ∂ is absorbing, meaning that $X_t \in \partial$ for all $t \geq \tau_\partial$, where $\tau_\partial = \inf\{t \geq 0 : X_t \in \partial\}$. We denote by $\mathbb{P}_x$ its law given $X_0 = x$ and we assume that the process X is absorbed in $\mathbb{P}_x$-a.s. finite time τ_∂ in ∂ for

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all $x \in E$. A quasi-stationary distribution for X is a probability measure ν_{QSD} on E such that

$$\mathbb{P}_{\nu_{QSD}}(X_t \in \cdot \mid t < \tau_\partial) = \nu_{QSD}, \quad \forall t \geq 0,$$

where $\mathbb{P}_\nu = \int_E \mathbb{P}_x \nu(dx)$ for all probability measure ν on E.

Our goal is to prove the existence and uniqueness of a quasi-stationary distribution for two models which are very standard in ecology and evolution (cf. e.g. [5, 6, 3] and the references therein). In fact, we prove the uniform convergence in total variation of the law of X_t given $X_t \notin \partial$ when $t \to +\infty$ to a unique quasi-stationary distribution. The first model is the multidimensional Lotka-Volterra birth and death process, and the second one the multidimensional competitive Lotka-Volterra (or logistic) Feller diffusion. These two models have received a lot of attention in the one-dimensional case, and a lot is known about their quasi-stationary behaviour [19, 2, 8], so we focus here on the multidimensional case, where the processes evolve on the state spaces $E \cup \partial = \mathbb{Z}_+^d$ for birth and death processes (with $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$) and $E \cup \partial = \mathbb{R}_+^d$ for diffusion processes, where $d \geq 2$. The case where $\partial = \{(0, \ldots, 0)\}$ can be handled combining the results of the present paper and those known in the one-dimensional case [9, 12, 8] following the methods of [3, Thm. 1.1]. So we focus here on the case where absorption corresponds to the extinction of a single population, instead of the extinction of the whole population. This case corresponds to $\partial = \mathbb{Z}_+^d \setminus \mathbb{N}^d$ and $E = \mathbb{N}^d$ (where $\mathbb{N} = \{1, 2, \ldots\}$) for multidimensional birth and death processes and $\partial = \mathbb{R}_+^d \setminus (0, +\infty)^d$ and $E = (0, +\infty)^d$ for multidimensional diffusions.

A general Lotka-Volterra birth and death process in dimension $d \geq 2$ is a Markov process $(X_t, t \geq 0)$ on $\mathbb{Z}_+^d$ with transition rates $q_{n,m}$ from $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ to $m \neq n$ in $\mathbb{Z}_+^d$ given by

$$q_{n,m} = \begin{cases} n_i(\lambda_i + \sum_{j=1}^d \gamma_{ij} n_j) & \text{if } m = n + e_i, \text{ for some } i \in \{1, \ldots, d\} \\ n_i(\mu_i + \sum_{j=1}^d c_{ij} n_j) & \text{if } m = n - e_i, \text{ for some } i \in \{1, \ldots, d\} \\ 0 & \text{otherwise,} \end{cases}$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is at the i-th coordinate. Note that $q_{n,n-e_i} = 0$ if $n_i = 0$, so that the process remains in the state space $\mathbb{Z}_+^d$. Since in addition $q_{n,m} = 0$ for all n such that $n_i = 0$ and m such that $m_i \geq 1$, the set $\partial = \mathbb{Z}_+^d \setminus \mathbb{N}^d$ is absorbing for the process. We make the usual convention that

$$q_{n,n} := -q_n := -\sum_{m \neq n} q_{n,m}.$$

From the biological point of view, the constant $\lambda_i \geq 0$ is the birth rate per individual of type $i \in \{1, \dots, d\}$, the constant $\mu_i \geq 0$ is the death rate per individual of type i, $c_{ij} \geq 0$ is the rate of death of an individual of type i from competition with an individual of type j, and $\gamma_{ij} \geq 0$ is the rate of birth of an individual of type i from cooperation with (or predation of) an individual of type j. In general, a Lotka-Volterra process could be explosive if some of the γ_{ij} are positive, but the assumptions of the next theorem ensure that it is not the case and that the process is almost surely absorbed in finite time.

Theorem 1.1. *Consider a competitive Lotka-Volterra birth and death process $(X_t, t \geq 0)$ in $\mathbb{Z}_+^d$ as above. Assume that the matrix $(c_{ij} - \gamma_{ij})_{1 \leq i,j \leq d}$ defines a positive operator on $\mathbb{R}_+^d$ in the sense that, for all $(x_1, \dots, x_d) \in \mathbb{R}_+^d \setminus \{0\}$, $\sum_{ij} x_i(c_{ij} - \gamma_{ij})x_j > 0$. Then the process has a unique quasi-stationary distribution ν_{QSD} and there exist constants $C, \gamma > 0$ such that, for all probability measures μ on $E = \mathbb{N}^d$,*

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq Ce^{-\lambda t}, \quad \forall t \geq 0, \tag{1.1}$$

where $\|\cdot\|_{TV}$ is the usual total variation distance defined by $\|\mu\|_{TV} = \sup_{f \in L^\infty(E),\ \|f\|_\infty \leq 1} |\mu(f)|$.

This result was previously known only in the competitive case (i.e. when $\gamma_{ij} = 0$ for all $1 \leq i, j \leq d$) and when the constants c_{ij} satisfy some bounds (see [10]). As explained in the last reference, an important difficulty of this problem is the fact that the absorption rate (i.e. the rate of jump from a state in E to a state in ∂) is not bounded with respect to the initial distribution. The existence of a quasi-stationary distribution for this kind of multi-dimensional birth and death processes can also be obtained using the theory of positive matrices, as exposed in [14], but the uniform exponential convergence (1.1) also implies uniqueness and several other properties, as recalled below.

A competitive Lotka-Volterra Feller diffusion process in dimension $d \geq 2$ is a Markov process $(X_t, t \geq 0)$ on $\mathbb{R}_+^d$, where $X_t = (X_t^1, \dots, X_t^d)$, is a solution of the stochastic differential equation

$$dX_t^i = \sqrt{\gamma_i X_t^i} dB_t^i + X_t^i \left(r_i - \sum_{j=1}^d c_{ij} X_t^j \right) dt, \quad \forall i \in \{1, \dots, d\}, \tag{1.2}$$

where $(B_t^1, t \geq 0), \dots, (B_t^d, t \geq 0)$ are independent standard Brownian motions. The Brownian terms and the linear drift terms correspond to classical

Feller diffusions, and the quadratic drift terms correspond to Lotka-Volterra interactions between coordinates of the process. The variances per individual γ_i is a positive number, and the growth rates per individual r_i can be any real number, for all $1 \leq i \leq d$. The competition parameters c_{ij} are assumed nonnegative for all $1 \leq i, j \leq d$, which corresponds to competitive Lotka-Volterra interaction. It is well known that, in this case, there is global strong existence and pathwise uniqueness for the SDE (1.2), and that it is almost surely absorbed in finite time in $\partial = \mathbb{R}_+^d \setminus (0, +\infty)^d$ if $c_{ii} > 0$ for all $i \in \{1, \dots, d\}$ (see [3] and Section 4).

Theorem 1.2. *Consider a competitive Lotka-Volterra Feller diffusion* $(X_t, t \geq 0)$ *in* $\mathbb{R}_+^d$ *as above. Assume that* $c_{ii} > 0$ *for all* $i \in \{1, \dots, d\}$. *Then the process has a unique quasi-stationary distribution* ν_{QSD} *and there exist constants* $C, \gamma > 0$ *such that, for all probability measures* μ *on* $E = (0, \infty)^d$,

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq Ce^{-\lambda t}, \quad \forall t \geq 0.$$

This results was previously known in dimension 2 when the constants c_{ij} and γ_i satisfy $c_{12}\gamma_1 = c_{21}\gamma_2$, which implies that the process (after some transformations) is a Kolmogorov diffusion (i.e. of the form $dY_t = dW_t - \nabla V(Y_t)dt$ for some Brownian motion W and some C^2 function V, see [3]). Our result is valid in any dimension and has no restriction on the coefficients. One can also expect to extend our result to cooperative cases (e.g. with $c_{21} < 0$ and $c_{12} < 0$ as in [3]) by using our abstract Lyapunov criterion with Lyapunov functions combining those used to prove Theorems 1.1 and 1.2.

To prove these two results, we make use of necessary and sufficient conditions for the convergence (1.1) obtained in [9]. These conditions are hard to check in practice (see e.g. [8, 4, 10]), so we first obtain in Section 2 several criteria based on generalized Lyapunov functions to check these conditions. These criteria apply to general Markov processes, not necessarily of the form of competitive Lotka-Volterra birth and death or diffusion processes. Our simplest Lyapunov criterion involves two bounded nonnegative functions V and γ such that $V(x)/\varphi(x) \to +\infty$ out of compact subsets of E such that

$$-L\varphi \leq C\mathbb{1}_K$$

for some compact subset K of E and

$$LV + a\frac{V^{1+\varepsilon}}{\varphi^\varepsilon} \leq b\varphi$$

for some $\varepsilon > 0$ and some constants $a > 0$ and $b \geq 0$, where L denotes (an extension of) the infinitesimal generator of the Markov process X. Together with additional conditions, the existence of such Lyapunov functions (V, φ) implies the convergence (1.1).

Many other properties can be deduced from (1.1). First, it implies the existence of a so-called *mortality/extinction plateau*: there exists $\lambda_0 > 0$ limit of the extinction rate of the population (see [20]). The constant $-\lambda_0$ is actually the largest non-trivial eigenvalue of the generator L and satisfies

$$\mathbb{P}_\alpha(t < \tau_\partial) = e^{-\lambda_0 t}, \quad \forall t \geq 0.$$

In addition, (1.1) implies that $x \mapsto e^{\lambda_0 t}\mathbb{P}_x(t < \tau_\partial)$ converges uniformly to a function $\eta : E \to (0, +\infty)$ when $t \to +\infty$ [11, Theorem 2.1]. Moreover, η is the eigenfunction of L corresponding to the eigenvalue $-\lambda_0$ [7, Prop. 2.3]. It also implies the existence and the exponential ergodicity of the associated Q-process, defined as the process X conditionned to never be extinct (see [7, Thm. 3.1] for a precise definition). The convergence of the conditional laws of X to the law of the Q-process holds also uniformly in total variation norm [11, Theorem 2.1], which entails conditional ergodic properties [11, Corollary 2.2].

Sections 3 and 4 are devoted to the study of (extensions of) competitive Lotka-Volterra birth and death processes and competitive Lotka-Volterra Feller diffusions and the proofs of Theorems 1.1 and 1.2.

2 General Lyapunov citeria for exponential convergence of conditional distributions

In order to obtain results as general as possible, we consider a general framework inspired from [21]. We first define this framework in Subsection 2.1, and we then state and prove a first version of our Lyapunov criterion in Subsection 2.2 and a second (simpler) version in Subsection 2.3.

2.1 Definitions and notations

We consider a right-continuous time-homogeneous Markov process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t, t \geq 0), (\mathbb{P}_x)_{x\in E\cup\{\partial\}})$ (see [22]) with state space $E \cup \partial$ such that $E \cap \partial = \emptyset$ where ∂ and E are topological spaces equiped with their Borel σ-fields. We assume that ∂ is an absorbing set for the process, which means that $X_t \in \partial$ for all $t \geq \tau_\partial$, where

$$\tau_\partial = \inf\{t \geq 0 : X_t \in \partial\}$$

is such that, for all $x \in E$ and $t \geq 0$, $\mathbb{P}_x(\tau_\partial < +\infty) = 1$ and $\mathbb{P}_x(t < \tau_\partial) > 0$.

Throughout the paper, we let $\{O_n, n \in \mathbb{N}\}$ and $\{U_n, n \in \mathbb{N}\}$ denote two fixed increasing families of subsets of E such that

$$E = \bigcup_{n\in\mathbb{N}} O_n \quad \text{and} \quad \text{cl}(O_n) \subset \text{int}(U_{n+1}) \subset \text{cl}(U_{n+1}) \subset \text{int}(O_{n+1}),\ \forall n \in \mathbb{N},$$

where $\text{cl}(A)$ and $\text{int}(A)$ denote respectively the closure and the interior in E of $A \subset E$. We assume that, for all $n \in \mathbb{N}$,

$$T_n := \inf\{t \geq 0 : X_t \notin O_n\}$$

is a $(\mathcal{F}_t)_{t\geq 0}$-stopping time and

$$\tau_{U_n} := \inf\{t \geq 0 : X_t \in U_n\}$$

is a $(\mathcal{F}_t)_{t\geq 0}$-stopping time to which the strong Markov property applies for $(X_t)_{t\geq 0}$. For example, the fact that T_n and τ_{U_n} are stopping times holds true when X is continuous, $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration, O_n are open sets and U_n are closed sets. This is also true if X is only right-continuous, $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, O_n are closed sets and U_n are open subsets. However, in the last case, the strong Markov property at time τ_{U_n} is harder to check. This is true for Feller processes, but for general stochastic differential equations, the well-posedness of the martingale problem only implies the strong Markov property on the natural filtration (see e.g. the books of Le Gall [17] or Rogers and Williams [22, 23]).

We assume that the process $(X_t, t \geq 0)$ is *regularly absorbed* in the sense that

$$\tau_\partial = \lim_{n\to+\infty} T_n, \quad \text{a.s. under } \mathbb{P}_x,\ \forall x \in E. \tag{2.1}$$

Thus, τ_∂ is a $(\mathcal{F}_t)_{t\geq 0}$-stopping time.

We also assume that the paths of $(X_t, t \geq 0)$ are càdlàg (right-continuous and admitting left limits) up to time T_n for all $n \in \mathbb{N}$. Note that the process X needs not be càdlàg since it may not admit a left limit at time $\tau_\partial = \lim_n T_n$.

We shall make use of the following weakened notion of generator for X.

Definition 2.1. *We say that a measurable function $V : E \cup \partial \to \mathbb{R}$ belongs to the domain $\mathcal{D}(L)$ of the weakened generator L of X if there exists a measurable function $W : E \to \mathbb{R}$ such that, for all $n \in \mathbb{N}$, $t \geq 0$ and $x \in E$,*

$$\mathbb{E}_x V(X_{t\wedge T_n}) = V(x) + \mathbb{E}_x\left[\int_0^{t\wedge T_n} W(X_s)ds\right] \ \textit{and}\ \mathbb{E}_x\left|\int_0^{t\wedge T_n} W(X_s)ds\right| < \infty, \tag{2.2}$$

and we define $LV = W$ on E. We also define $LV(x) = 0$ for all $x \in \partial$.

So far, the operator L may be multi-valued, but it is enough for us to define LV as any W satisfying the above property. Of course, this definition extends the usual definition of the extended generator of X, and therefore of the usual weak and infinitesimal generators (cf. [13, 21]).

We introduce the following notation.

Definition 2.2. *We say that a function $f : E \to \mathbb{R}$ converges to a limit ℓ out of $(O_n)_{n\geq 1}$ and we write*

$$\lim_{x\notin O_n,\ n\to\infty} f(x) = \ell$$

if, for all sequence $(x_k)_{k\geq 1}$ in E such that $\{k \in \mathbb{N} : x_k \in O_n\}$ is finite for all $n \geq 1$,

$$\lim_{k\to\infty} f(x_k) = \ell.$$

Equivalently, this means that

$$\lim_{n\to+\infty} \sup_{x\in E\setminus O_n} |f(x) - \ell| = 0.$$

We also define the set of admissible functions to which our Lyapunov criteria apply.

Definition 2.3. *We say that a couple (V, φ) of functions V and φ measurable from $E \cup \partial$ to $\mathbb{R}$ is an* admissible couple of functions *if*

(i) *V and φ are bounded nonnegative on $E \cup \partial$ and positive on E, $V(x) = \varphi(x) = 0$ for all $x \in \partial$ and*

$$\inf_{x\in E} \frac{V(x)}{\varphi(x)} > 0. \tag{2.3}$$

(ii) *We have the convergences*

$$\lim_{x\notin O_n,\ n\to\infty} \frac{V(x)}{\varphi(x)} = +\infty \tag{2.4}$$

and

$$\lim_{n\to+\infty} V(X_{T_n}) = 0 \quad a.s. \tag{2.5}$$

(iii) *V and φ belong to the domain of the weakened generator L of X, LV is bounded from above and $L\varphi$ is bounded from below.*

Note that since $V(X_t) = V(X_t)\mathbb{1}_{t<\tau_\partial}$, (2.5) is actually equivalent to

$$\lim_{n\to+\infty} V(X_{t\wedge T_n}) = V(X_t), \quad \forall t \geq 0, \text{ a.s.} \tag{2.6}$$

Moreover, since V is bounded, (2.4) implies that

$$\lim_{x\notin O_n,\ n\to\infty} \varphi(x) = 0.$$

Therefore, using the fact that $X_{T_{n+1}} \notin O_n$ for all $n \geq 1$ since X is right continuous, an admissible couple of functions (V, φ) also satisfies

$$\lim_{n\to+\infty} \varphi(X_{t\wedge T_n}) = \varphi(X_t), \quad \forall t \geq 0, \text{ a.s.} \tag{2.7}$$

2.2 A non-linear Lyapunov criterion

Given a couple of admissible functions (V, φ), our first Lyapunov criterion is based on the following assumption.

Assumption 1. *The couple of admissible functions (V, φ) satisfies that there exist constants $\varepsilon, A, B > 0$ such that, for all probability measure μ on E,*

$$\mu(LV) - \mu(V)\frac{\mu(L\varphi)}{\mu(\varphi)} \leq A\mu(\varphi) - B\frac{\mu(V)^{1+\varepsilon}}{\mu(\varphi)^\varepsilon} \tag{2.8}$$

and there exist constants $r_0, p_0 > 0$ such that, for n large enough,

$$\mathbb{P}_x(r_0 < \tau_\partial) \leq p_0 V(x), \quad \forall x \in E \setminus O_n. \tag{2.9}$$

Assumption (2.8) can be seen as a nonlinear Lyapunov criterion. We also need to assume a local Doeblin property.

Assumption 2. *There exist $k_0 \in \mathbb{N}$, $\theta_0, \theta_1, a_1 > 0$ and a probability measure ν on E such that, for all $x \in O_{k_0}$ and all $s \in [\theta_1, \theta_1 + \theta_0]$,*

$$\mathbb{P}_x(X_s \in \cdot) \geq a_1 \nu.$$

In addition, we assume the following weak irreducibility: for all $n \geq k_0$, there exists $s_n \geq 0$ such that

$$D_n := \inf_{x\in O_n} \mathbb{P}_x(X_{s_n} \in O_{k_0}) > 0.$$

The following assumption is about the uniform boundedness of exponential moments of the hitting time of the sets $U_n \cup \partial$.

Assumption 3. *For all $\lambda > 0$, there exists $n \geq 1$ such that*

$$\sup_{x\in E} \mathbb{E}_x(e^{\lambda(\tau_{U_n}\wedge\tau_\partial)}) < \infty. \tag{2.10}$$

Our last assumption is a form of local Harnack inequality, uniform in time.

Assumption 4. *For all $n \geq 0$, there exists a constant C_n such that, for all $t \geq 0$,*

$$\sup_{x\in O_n} \mathbb{P}_x(t < \tau_\partial) \leq C_n \inf_{x\in O_n} \mathbb{P}_x(t < \tau_\partial). \tag{2.11}$$

We are now able to state our first result, which will be applied in the subsequent sections.

Theorem 2.4. *Assume that the process $(X_t, t \geq 0)$ is regularly absorbed and that there exists a couple of admissible functions (V, φ) satisfying Assumption 1. Assume also that Assumptions 2, 3 and 4 are satisfied. Then the process X admits a unique quasi-stationary distribution ν_{QSD} and there exist constants $C, \gamma > 0$ such that for all probability measure μ on E,*

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq Ce^{-\gamma t}, \quad \forall t \geq 0. \tag{2.12}$$

The proof of Theorem 2.4 is based on the next result.

Proposition 2.5. *Assume that the process $(X_t, t \geq 0)$ is regularly absorbed and that there exists a couple of admissible functions (V, φ) satisfying Assumption 1. Then, for all $x \in E$ and $t \geq 0$,*

$$\frac{\mathbb{E}_x[V(X_t)]}{\mathbb{E}_x[\varphi(X_t)]} = \frac{V(x)}{\varphi(x)} + \int_0^t \left\{ \frac{\mathbb{E}_x[LV(X_s)]}{\mathbb{E}_x[\varphi(X_s)]} - \frac{\mathbb{E}_x[V(X_s)]}{\mathbb{E}_x[\varphi(X_s)]} \frac{\mathbb{E}_x[L\varphi(X_s)]}{\mathbb{E}_x[\varphi(X_s)]} \right\} ds, \tag{2.13}$$

where the value of the integral in the r.h.s. is well-defined since, for all $t \geq 0$,

$$\frac{\mathbb{E}_x[LV(X_s)]}{\mathbb{E}_x[\varphi(X_s)]} - \frac{\mathbb{E}_x[V(X_s)]}{\mathbb{E}_x[\varphi(X_s)]} \frac{\mathbb{E}_x[L\varphi(X_s)]}{\mathbb{E}_x[\varphi(X_s)]} \in L^1([0,t]).$$

Proof. Using the Definition 2.1 of the weakened infinitesimal generator, we have for all $n \geq 1$

$$\mathbb{E}_x V(X_{t\wedge T_n}) = V(x) + \mathbb{E}_x \int_0^{t\wedge T_n} LV(X_s)ds = V(x) + \mathbb{E}_x \int_0^t LV(X_s)\mathbb{1}_{s<T_n}ds. \tag{2.14}$$

Applying (2.6) and Lebesgue's theorem to the left-hand side and Fatou's lemma to the right-hand side (using that LV is bounded from above), we obtain

$$\mathbb{E}_x V(X_t) \leq V(x) + \mathbb{E}_x \left[\int_0^{t\wedge\tau_\partial} LV(X_s) \right] ds = V(x) + \int_0^t \mathbb{E}_x \left[LV(X_s) \right] ds.$$

Since $V \geq 0$ and LV is bounded from above, we deduce that $\mathbb{E}_x LV(X_s) \in L^1([0,t])$ and $LV(X_s) \in L^1(\Omega \times [0,t])$. Therefore, we can actually apply Lebesgue's Theorem to the right-hand side of (2.14) and hence

$$\mathbb{E}_x V(X_t) = V(x) + \int_0^t \mathbb{E}_x \left[LV(X_s) \right] ds.$$

The same argument applies to $-\varphi$ using (2.7):

$$\mathbb{E}_x \varphi(X_t) = \varphi(x) + \int_0^t \mathbb{E}_x \left[L\varphi(X_s) \right] ds. \tag{2.15}$$

Therefore (cf. e.g. [1, Thm. VIII.2 and Lem. VIII.2]), for all $T > 0$, $t \mapsto (\mathbb{E}_x V(X_t), \mathbb{E}_x \varphi(X_t))$ belongs to the Sobolev space $W^{1,1}([0,T], \mathbb{R}^2)$ (the set of functions from $[0,T]$ to $\mathbb{R}^2$ in L^1 admitting a derivative in the sense of distributions in L^1).

In particular, $t \mapsto \mathbb{E}_x \varphi(X_t)$ is continuous on $[0,T]$. Since $\mathbb{P}_x(t < \tau_\partial) > 0$ and hence $\mathbb{E}_x \varphi(X_t) > 0$ for all $t \in [0,T]$, we deduce that $\inf_{t\in[0,T]} \mathbb{E}_x \varphi(X_t) > 0$. Therefore, we deduce from standard properties of $W^{1,1}$ functions [1, Cor. VIII.9 and Cor. VIII.10] that $t \mapsto \mathbb{E}_x V(X_t)/\mathbb{E}_x \varphi(X_t)$ also belongs to $W^{1,1}$ and admits as derivative

$$t \mapsto \frac{\mathbb{E}_x LV(X_t)}{\mathbb{E}_x \varphi(X_t)} - \frac{\mathbb{E}_x V(X_t)\, \mathbb{E}_x L\varphi(X_t)}{\mathbb{E}_x [\varphi(X_t)]^2} \in L^1([0,T]).$$

Hence we have proved (2.13). □

Before proving Theorem 2.4, let us prove that Assumption 2 is equivalent to a seemingly stronger assumption.

Lemma 2.6. *Assumption 2 implies that, for all $n_0 \geq k_0$ and for all $t_0 > 0$, there exist $t_1, c_1 > 0$ such that, for all $s \in [t_1, t_1 + t_0]$ and $x \in O_{n_0}$,*

$$\mathbb{P}_x(X_s \in \cdot) \geq c_1 \nu. \tag{2.16}$$

In addition, for all $n \geq n_0 \geq k_0$,

$$\inf_{x\in O_n} \mathbb{P}_x(X_{s_n} \in O_{n_0}) \geq D_n > 0. \tag{2.17}$$

Proof. Equation (2.17) is an immediate consequence of the definition of D_n and the fact that $O_{k_0} \subset O_{n_0}$.

To prove (2.16), let $m_0 \geq k_0$ be such that $\nu(O_{m_0}) > 0$. Then, for all $x \in O_{m_0}$ and all $s \in [s_{m_0} + \theta_1, s_{m_0} + \theta_1 + \theta_0]$,

$$\mathbb{P}_x(X_s \in \cdot) \geq D_{m_0} a_1 \nu(\cdot).$$

Hence, for all $\ell \in \mathbb{N}$, $x \in O_{m_0}$ and $s \in [\ell(s_{m_0} + \theta_1), \ell(s_{m_0} + \theta_1 + \theta_0)]$,

$$\mathbb{P}_x(X_s \in \cdot) \geq (D_{m_0} a_1)^\ell \, \nu(O_{m_0})^{\ell-1} \nu(\cdot).$$

Therefore, for all $n_0 \geq k_0$, $x \in O_{n_0}$ and $s \in [s_{n_0} + \ell(s_{m_0} + \theta_1), s_{n_0} + \ell(s_{m_0} + \theta_1 + \theta_0)]$,

$$\mathbb{P}_x(X_s \in \cdot) \geq D_{n_0} \, (D_{m_0} a_1)^\ell \, \nu(O_{m_0})^{\ell-1} \nu(\cdot).$$

This ends the proof of Lemma 2.6. □

Proof of Theorem 2.4. The convergence (2.12) can be proved using the necessary and sufficient criterion obtained in [7] for general Markov processes. This criterion is given by the two conditions (A1) and (A2) below.

There exists a probability measure ν on E such that

(A1) there exist $t'_0, c'_1 > 0$ such that for all $x \in E$,

$$\mathbb{P}_x(X_{t'_0} \in \cdot \mid t'_0 < \tau_\partial) \geq c'_1 \nu(\cdot);$$

(A2) there exists $c'_2 > 0$ such that for all $x \in E$ and $t \geq 0$,

$$\mathbb{P}_\nu(t < \tau_\partial) \geq c'_2 \mathbb{P}_x(t < \tau_\partial).$$

The proof is divided in two steps, where (A1) and (A2) are proved respectively.

Step 1: Proof of (A1).
Assumption (2.8) and Proposition 2.5 imply that

$$\frac{\mathbb{E}_x V(X_t)}{\mathbb{E}_x \varphi(X_t)} \leq \frac{V(x)}{\varphi(x)} + At - B \int_0^t \left(\frac{\mathbb{E}_x V(X_s)}{\mathbb{E}_x \varphi(X_s)} \right)^{1+\varepsilon} ds.$$

Fix $t > 0$ and assume that for all $s \in [0, t]$, $\mathbb{E}_x V(X_s)/\mathbb{E}_x \varphi(X_s) \geq a$, where $a = (2A/B)^{1/(1+\varepsilon)}$. Then

$$\frac{\mathbb{E}_x V(X_t)}{\mathbb{E}_x \varphi(X_t)} \leq \frac{V(x)}{\varphi(x)} - \frac{B}{2} \int_0^t \left(\frac{\mathbb{E}_x V(X_s)}{\mathbb{E}_x \varphi(X_s)} \right)^{1+\varepsilon} ds.$$

Integrating this differential inequality entails

$$\frac{\mathbb{E}_x V(X_t)}{\mathbb{E}_x \varphi(X_t)} \leq \left[\left(\frac{\varphi(x)}{V(x)}\right)^{\varepsilon} + \frac{\varepsilon B t}{2}\right]^{-1/\varepsilon} \leq \left(\frac{2}{\varepsilon B t}\right)^{1/\varepsilon}.$$

Choosing $t = t_0$ such that $\left(\frac{2}{\varepsilon B t_0}\right)^{1/\varepsilon} < a$ gives a contradiction, hence there exists $u_x \in [0, t_0]$ such that $\mathbb{E}_x V(X_{u_x})/\mathbb{E}_x \varphi(X_{u_x}) < a$. Note that the value of u_x may depend on $x \in E$ but the value of t_0 is independent of $x \in E$. We also define n_0 as the first integer n such that

$$\inf_{y \in E \setminus O_n} \frac{V(y)}{\varphi(y)} \geq 2a,$$

whose existence is ensured by Assumption (2.4).

Now,

$$\begin{aligned}
a &\geq \frac{\mathbb{E}_x V(X_{u_x})}{\mathbb{E}_x \varphi(X_{u_x})} \\
&\geq \frac{\mathbb{E}_x[V(X_{u_x}) \mathbb{1}_{X_{u_x} \in E \setminus O_{n_0}}] + \mathbb{E}_x[V(X_{u_x}) \mathbb{1}_{X_{u_x} \in O_{n_0}}]}{\sup_{y \in E \setminus O_{n_0}} \frac{\varphi(y)}{V(y)} \mathbb{E}_x[V(X_{u_x}) \mathbb{1}_{X_{u_x} \in E \setminus O_{n_0}}] + \sup_{y \in E} \frac{\varphi(y)}{V(y)} \mathbb{E}_x[V(X_{u_x}) \mathbb{1}_{X_{u_x} \in O_{n_0}}]} \\
&\geq \frac{\mathbb{E}_x[V(X_{u_x}) \mathbb{1}_{X_{u_x} \in E \setminus O_{n_0}}] + \mathbb{E}_x[V(X_{u_x}) \mathbb{1}_{X_{u_x} \in O_{n_0}}]}{\frac{1}{2a} \mathbb{E}_x[V(X_{u_x}) \mathbb{1}_{X_{u_x} \in E \setminus O_{n_0}}] + \sup_{y \in E} \frac{\varphi(y)}{V(y)} \mathbb{E}_x[V(X_{u_x}) \mathbb{1}_{X_{u_x} \in O_{n_0}}]}.
\end{aligned}$$

Since $a > \mathbb{E}_x V(X_{u_x})/\mathbb{E}_x \varphi(X_{u_x}) \geq 1/\sup_{y \in E}(\varphi(y)/V(y))$, we deduce that

$$\mathbb{E}_x[V(X_{u_x}) \mathbb{1}_{X_{u_x} \in O_{n_0}}] \geq \frac{1}{2(a \sup_{y \in E} \frac{\varphi(y)}{V(y)} - 1)} \mathbb{E}_x[V(X_{u_x}) \mathbb{1}_{X_{u_x} \in E \setminus O_{n_0}}].$$

On the one hand, this last inequality, Assumption (2.9) and Markov's property entails (where s_{n_0} is taken from (2.17) of Lemma 2.6)

$$\begin{aligned}
\mathbb{P}_x\left(u_x + s_{n_0}\left\lceil \frac{r_0}{s_{n_0}} \right\rceil < \tau_\partial\right) &\leq \mathbb{P}_x(u_x + r_0 < \tau_\partial) \\
&\leq \mathbb{P}_x(X_{u_x} \in O_{n_0}) + \mathbb{E}_x[\mathbb{P}_{X_{u_x}}(r_0 < \tau_\partial) \mathbb{1}_{X_{u_x} \in E \setminus O_{n_0}}] \\
&\leq \mathbb{P}_x(X_{u_x} \in O_{n_0}) + p_0 \mathbb{E}_x[V(X_{u_x}) \mathbb{1}_{X_{u_x} \in E \setminus O_{n_0}}] \\
&\leq \mathbb{P}_x(X_{u_x} \in O_{n_0}) + 2p_0\left(a \sup_{y \in E} \frac{\varphi(y)}{V(y)} - 1\right) \mathbb{E}_x[V(X_{u_x}) \mathbb{1}_{X_{u_x} \in O_{n_0}}] \\
&\leq \mathbb{P}_x(X_{u_x} \in O_{n_0}) \left[1 + 2p_0 \|V\|_\infty \left(a \sup_{y \in E} \frac{\varphi(y)}{V(y)} - 1\right)\right].
\end{aligned}$$

On the other hand, using (2.17) and Markov's property, for all $x \in E$, and all $k \geq 1$,

$$\mathbb{P}_x(X_{u_x+ks_{n_0}} \in O_{n_0}) \geq \mathbb{P}_x(X_{u_x} \in O_{n_0}) \inf_{y \in O_{n_0}} \mathbb{P}_y(X_{ks_{n_0}} \in O_{n_0}) \geq D_{n_0}^k \mathbb{P}_x(X_{u_x} \in O_{n_0}).$$

Therefore, choosing $k = \lceil \frac{r_0}{s_{n_0}} \rceil$, we have proved that

$$\mathbb{P}_x(X_{u_x+ks_{n_0}} \in O_{n_0} \mid u_x + ks_{n_0} < \tau_\partial) \geq \frac{D_{n_0}^k}{1 + 2p_0 \|V\|_\infty \left(a \sup_{y \in E} \varphi(y)/V(y) - 1\right)} =: q.$$

Now, since $u_x \in [0, t_0]$, Equation (2.16) of Lemma 2.6 entails $\mathbb{P}_y(X_{t_1+t_0-u_x} \in \Gamma) \geq c_1 \nu(\Gamma)$ for all $y \in O_{n_0}$ and all $\Gamma \subset E$ measurable. Hence, denoting $v_x = u_x + ks_{n_0}$,

$$\begin{aligned}\mathbb{P}_x(X_{t_1+t_0+ks_{n_0}} \in \Gamma \mid t_1 + t_0 + ks_{n_0} < \tau_\partial) &= \frac{\mathbb{E}_x\left[\mathbb{1}_{v_x < \tau_\partial} \mathbb{P}_{X_{v_x}}(X_{t_1+t_0-u_x} \in \Gamma)\right]}{\mathbb{P}_x(t_1 + t_0 + ks_{n_0} < \tau_\partial)} \\ &\geq c_1 \nu(\Gamma) \frac{\mathbb{P}_x(X_{v_x} \in O_{n_0})}{\mathbb{P}_x(t_1 + t_0 + ks_{n_0} < \tau_\partial)} \\ &\geq c_1 \nu(\Gamma) \mathbb{P}_x(X_{v_x} \in O_{n_0} \mid v_x < \tau_\partial) \\ &\geq c_1 q \nu(\Gamma).\end{aligned}$$

This concludes the proof of (A1), with $c_1' = c_1 q$ and $t_1' = t_1 + t_0 + ks_{n_0}$.

Step 2: Proof of (A2).
Since $\cup_n O_n = E$, we can fix $n_1 \geq n_0$ such that $\nu(O_{n_1}) \geq 1/2$. Let $m \geq n_0$ and $0 \leq s \leq t$, and define ℓ the smallest integer such that $s \leq s_m + \ell(t_1' + s_{n_1})$.

Using (2.17), we obtain, for all $x \in O_m$,

$$\begin{aligned}
&\mathbb{P}_x(t < \tau_\partial) \\
&\geq \mathbb{P}_x(X_{s_m} \in O_{n_0},\, X_{s_m+t'_1} \in O_{n_1},\, X_{s_m+t'_1+s_{n_1}} \in O_{n_0},\, X_{s_m+2t'_1+s_{n_1}} \in O_{n_1}, \\
&\qquad \ldots, X_{s_m+\ell(t'_1+s_{n_1})} \in O_{n_0},\, t - s + s_m + \ell(t'_1 + s_{n_1}) < \tau_\partial) \\
&\geq D_m \left(\frac{c'_1}{2} D_{n_1}\right)^{\ell} \inf_{x\in O_{n_0}} \mathbb{P}_x(t - s < \tau_\partial) \\
&\geq D_m \left(\frac{c'_1}{2} D_{n_1}\right)^{1+s/(t'_1+s_{n_1})} \inf_{x\in O_m} \mathbb{P}_x(t - s < \tau_\partial) \\
&\geq \frac{c'_1 D_m D_{n_1}}{2} \inf_{x\in O_m} \mathbb{P}_x(t - s < \tau_\partial) \exp(-\lambda s),
\end{aligned}$$

where

$$\lambda := \frac{1}{t'_1 + s_{n_1}} \log\left(\frac{2}{c'_1 D_{n_1}}\right) > 0$$

is independent of $m \geq n_0$ and $x \in O_m$. Using Assumption 4, we obtain for all $0 \leq s \leq t$

$$\inf_{x\in O_m} \mathbb{P}_x(t < \tau_\partial) \geq \frac{c_1 D_m D_{n_1}}{2C_m} \sup_{x\in O_m} \mathbb{P}_x(t - s < \tau_\partial) \exp(-\lambda s). \tag{2.18}$$

Now, we apply Assumption 3 for λ as defined above: there exists $m \geq n_0 \vee n_1$ such that

$$M := \sup_{x\in E} \mathbb{E}_x[\exp(\lambda(\tau_{U_m} \wedge \tau_\partial))] < \infty. \tag{2.19}$$

Recall that we assumed that the strong Markov property applies at the stopping time τ_{U_m}. Note also that $X_{\tau_{U_m}} \in O_m$ on the event $\{\tau_{U_m} < \infty\}$. Hence, using (2.19) and the strong Markov property,

$$\begin{aligned}
\mathbb{P}_x(t < \tau_\partial) &= \mathbb{P}_x(t < \tau_{U_m} \wedge \tau_\partial) + \mathbb{P}_x(\tau_{U_m} \leq t < \tau_\partial) \\
&\leq M e^{-\lambda t} + \int_0^t \sup_{y\in O_m} \mathbb{P}_y(t - s < \tau_\partial) \mathbb{P}_y(\tau_{U_m} \in ds).
\end{aligned}$$

Now, for all $s \leq t$, $\mathbb{P}_y(\tau_{U_m} \in ds) \leq \mathbb{P}_y(\tau_{U_m} \wedge \tau_\partial \in ds)$. Hence, using (2.18) twice and the fact that $\nu(O_{n_1}) \geq 1/2$, we have for all $x \in E$

$$\begin{aligned}\mathbb{P}_x(t < \tau_\partial) &\leq Me^{-\lambda t} + \int_0^t \sup_{y\in O_m} \mathbb{P}_y(t-s < \tau_\partial)\mathbb{P}_y(\tau_{U_m} \wedge \tau_\partial \in ds) \\ &\leq \frac{2C_m}{c_1 D_m D_{n_1}} \Bigg[M \inf_{y\in O_m} \mathbb{P}_y(t < \tau_\partial) \\ &\qquad + \inf_{y\in O_m} \mathbb{P}_y(t<\tau_\partial) \int_0^t e^{\lambda s}\mathbb{P}_x(\tau_{U_m}\wedge\tau_\partial \in ds)\Bigg] \\ &\leq \frac{4MC_m}{c_1 D_m D_{n_1}} \inf_{y\in O_{n_1}} \mathbb{P}_y(t<\tau_\partial) \\ &\leq \frac{8MC_m}{c_1 D_m D_{n_1}} \mathbb{P}_\nu(t<\tau_\partial).\end{aligned}$$

This is (A2). □

2.3 A simpler Lyapunov criterion

Condition (2.8) may be hard to check in practice, so we give a stronger condition, easier to check.

Proposition 2.7. *Assume that (V, φ) is a couple of admissible functions such that $\inf_{x\in O_m} \varphi(x) > 0$ for all integer m. Then Condition (2.8) is implied by the following two conditions:*

(a) *There exist an integer n and a constant $C \geq 0$ such that*

$$-L\varphi \leq C\mathbb{1}_{O_n}. \tag{2.20}$$

(b) *There exist constants $C' > 0$ and $C'' \geq 0$ such that*

$$LV + C'\frac{V^{1+\varepsilon}}{\varphi^\varepsilon} \leq C''\varphi. \tag{2.21}$$

Note that Conditions (a) and (b) imply that LV is bounded from above and $L\varphi$ is bounded from below (which are parts of the conditions for a couple of functions to be admissible). This result immediately entails the following corollary, which will be used in the applications provided in Sections 3 and 4.

Corollary 2.8. *Assume that the process $(X_t, t \geq 0)$ is regularly absorbed and that there exists a couple of admissible functions (V, φ) satisfying (2.9) and Conditions (a) and (b) of Proposition 2.7 and such that $\inf_{x\in O_m} \varphi(x) > 0$*

for all integer m. Assume also that Assumptions 2, 3 and 4 are satisfied. Then the process X admits a unique quasi-stationary distribution ν_{QSD} and there exist constants $C, \gamma > 0$ such that, for all probability measure μ on E,

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq C e^{-\gamma t}, \quad \forall t \geq 0.$$

Proof of Proposition 2.7. Let μ be any probability measure on E. On the one hand, it follows from (2.20) that

$$-\frac{\mu(L\varphi)}{\mu(\varphi)} \leq \frac{C}{\inf_{x \in O_n} \varphi(x)}.$$

On the other hand, Hölder's inequality with $p = (1+\varepsilon)/\varepsilon$ and $q = 1+\varepsilon$ yields

$$\begin{aligned} \mu(V)^{1+\varepsilon} &\leq \mu\left(\frac{V^q}{\varphi^{\varepsilon q/(1+\varepsilon)}}\right)^{(1+\varepsilon)/q} \mu(\varphi^{\varepsilon p/(1+\varepsilon)})^{(1+\varepsilon)/p} \\ &= \mu\left(\frac{V^{1+\varepsilon}}{\varphi^\varepsilon}\right) \mu(\varphi)^\varepsilon. \end{aligned}$$

Hence we deduce from (2.21) that

$$\begin{aligned} &\mu(LV) - \frac{\mu(L\varphi)}{\mu(\varphi)} \mu(V) \\ &\qquad \leq C''\mu(\varphi) - C'\mu\left(\frac{V^{1+\varepsilon}}{\varphi^\varepsilon}\right) + \frac{C}{\inf_{x \in O_n} \varphi(x)} \mu(V) \\ &\qquad \leq C''\mu(\varphi) + \mu\left[V\left(\frac{C}{\inf_{x \in O_n} \varphi(x)} - \frac{C'V^\varepsilon}{2\varphi^\varepsilon}\right)\right] - \frac{C'}{2} \frac{\mu(V)^{1+\varepsilon}}{\mu(\varphi)^\varepsilon}. \end{aligned}$$

Now, because of Assumption (2.4), there exists m large enough such that, for all $x \in E \setminus O_m$,

$$\frac{C'V^\varepsilon(x)}{2\varphi^\varepsilon(x)} \geq \frac{C}{\inf_{y \in O_n} \varphi(y)}.$$

Therefore,

$$\begin{aligned} &\mu(LV) - \frac{\mu(L\varphi)}{\mu(\varphi)} \mu(V) \\ &\qquad \leq C''\mu(\varphi) + \frac{C}{\inf_{x \in O_n} \varphi(x)} \mu(V \mathbb{1}_{O_m}) - \frac{C'}{2} \frac{\mu(V)^{1+\varepsilon}}{2\mu(\varphi)^\varepsilon} \\ &\qquad \leq \left(C'' + \frac{C \sup_{x \in E} V(x)}{\inf_{x \in O_n} \varphi(x) \inf_{x \in O_m} \varphi(x)}\right) \mu(\varphi) - \frac{C'}{2} \frac{\mu(V)^{1+\varepsilon}}{2\mu(\varphi)^\varepsilon}. \end{aligned}$$

This is (2.8). □

3 Application to multidimensional birth and death processes absorbed when one of the coordinates hits 0

We consider general multitype birth and death processes in continuous time, taking values in $\mathbb{Z}_+^d$ for some $d \geq 2$. Let $(X_t, t \geq 0)$ be a Markov process on $\mathbb{Z}_+^d$ with transition rates

$$\text{from } n = (n_1, \ldots, n_d) \text{ to } \begin{cases} n + e_j & \text{with rate } n_j b_j(n), \\ n - e_j & \text{with rate } n_j d_j(n) \end{cases}$$

for all $1 \leq j \leq d$, with $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the nonzero coordinate is the j-th one, $b(n) = (b_1(n), \ldots, b_r(n))$ and $d(n) = (d_1(n), \ldots, d_r(n))$ are functions from $\mathbb{Z}_+^d$ to $\mathbb{R}_+^d$. This model represents a density-dependent population dynamics with d types of individuals (say d species), where $b_i(n)$ (resp. $d_i(n)$) represents the reproduction rate (resp. death rate) per individuals of species i when the population is in state n.

Note that the forms of the birth and death rates imply that, once a coordinate X_t^j of the process hits 0, it remains equal to 0. This corresponds to the extinction of the population of type j. Hence, the set $\partial := \mathbb{Z}_+^r \setminus \mathbb{N}^r$ is absorbing for the process X and we denote as usual by τ_∂ its absorption time.

We define for all $1 \leq k \leq d$

$$\bar{d}(k) = \sup_{n \in \mathbb{N}^d,\ |n| = k} |n| \sum_{i=1}^{d} \mathbb{1}_{n_i = 1} d_i(n),$$

$$\text{and} \quad \underline{d}(k) = \inf_{n \in \mathbb{N}^d,\ |n| = k} \sum_{i=1}^{d} n_i \left[\mathbb{1}_{n_i \neq 1} d_i(n) - b_i(n) \right],$$

where $|n| := n_1 + \ldots + n_d$. We shall assume

Assumption 5. *There exists $\eta > 0$ small enough such that, for all $k \in \mathbb{N}$ large enough,*

$$\underline{d}(k) \geq \eta \bar{d}(k), \tag{3.1}$$

and

$$\frac{\underline{d}(k)}{k^{1+\eta}} \xrightarrow[k \to +\infty]{} +\infty. \tag{3.2}$$

Note that, assuming that the set O_n is finite for all n, any function $f : \mathbb{Z}_+^d \to \mathbb{R}$ is in the domain of the weakened infinitesimal generator of X and, for all $n \in \mathbb{N}_+^d$,

$$Lf(n) = \sum_{j=1}^{d} [f(n+e_j) - f(n)] n_j b_j(n) + \sum_{j=1}^{d} [f(n-e_j) - f(n)] n_j d_j(n).$$

Under Assumption (3.2), setting $W(n) = |n|$, we have

$$\frac{LW(n)}{W(n)^{1+\eta}} \to -\infty.$$

This classically entails that

$$\sup_{n \in \mathbb{N}^r} \mathbb{E}_n(|X_1|) < +\infty. \tag{3.3}$$

(The argument is very similar to the one used for Theorem 2.4.) In particular, the process is non-explosive and τ_∂ is finite almost surely. Therefore, if we define for all $m \geq 1$

$$O_m = U_m = \{x \in \mathbb{N}^d : |x| \leq m\},$$

it is clear that the process X satisfies the conditions of Section 2.1 and is regularly absorbed. We can now state the main result of the section.

Theorem 3.1. *Under Assumption 5, the multi-dimensional competitive birth and death process $(X_t, t \geq 0)$ absorbed when one of its coordinates hits 0 admits a unique quasi-stationnary distribution ν_{QSD} and there exist constants $C, \gamma > 0$ such that, for all probability measure μ on $\mathbb{N}^d$,*

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq Ce^{-\gamma t}, \quad \forall t \geq 0.$$

It is easy to check that Assumption 5 is satisfied in the general Lotka-Volterra birth and death process of the introduction. Indeed, we clearly have $\bar{d}(k) \leq Ck^2$ and

$$\begin{aligned}\underline{d}(k) &\geq \inf_{n \in \mathbb{N}^d,\ |n|=k} \sum_{i=1}^{d} (d_i(n) - b_i(n)) - \sup_{n \in \mathbb{N}^d,\ |n|=k} \sum_{i=1}^{n} d_i(n) \mathbb{1}_{n_i=1} \\ &\geq -Ck + \inf_{n \in \mathbb{N}^d,\ |n|=k} \sum_{i,j=1}^{d} n_i (c_{ij} - \gamma_{ij}) n_j\end{aligned}$$

for $C = \sup_i(b_i + d_i) + \sup_{ij} c_{ij}$. Under the assumptions of Theorem 1.1, there exists $C' > 0$ such that, for all $n \in \mathbb{N}^d$,

$$\sum_{i,j=1}^{d} n_i(c_{ij} - \gamma_{ij})n_j \geq C'|n|^2.$$

This entails Assumption 5.

Proof. We are going to use the criterion of Corollary 2.8. Using (3.3) and copying the arguments of [9, Sec. 4.1.1 and Thm. 4.1], one deduces that Assumption 2 is satisfied with $\nu = \delta_{(1,\dots,1)}$ and Assumptions 3 and 4 are also satisfied.

Hence we only have to find a couple of admissible functions (V, φ) satisfying Conditions (a) and (b) of Proposition 2.7 and satisfying (2.9) (note that the fact that φ is positive on $\mathbb{N}^d$ entails immediately that $\inf_{x\in O_m} \varphi(x) > 0$). This couple of functions is given for all $n \in \mathbb{Z}_+^d$ by

$$V(n) = \begin{cases} \sum_{k=1}^{|n|} \frac{1}{k^\alpha} & \text{if } n \in \mathbb{N}^d, \\ 0 & \text{if } n \in \partial \end{cases}$$

and

$$\varphi(n) = \begin{cases} \sum_{k=|n|+1}^{+\infty} \frac{1}{k^\beta} & \text{if } n \in \mathbb{N}^d, \\ 0 & \text{if } n \in \partial, \end{cases}$$

for appropriate choices of $\alpha, \beta > 1$. Note that the two functions are bounded, nonnegative and positive of $\mathbb{N}^d$. Note also that, since $\inf_{n\in\mathbb{N}^d} V(n) > 0$, Conditions (2.3) and (2.9) are obviously satisfied. Note also that (2.5) is clear for non-explosive Markov processes on discrete state spaces, and that (2.4) is true from the definition of V and φ.

Hence, we only have to check Conditions (a) and (b) since they imply the last condition needed for the couple of functions (V, φ) to be admissible: the fact that LV is bounded from above and $L\varphi$ bounded from below. So we compute

$$\begin{aligned} L\varphi(n) &= -\sum_{i=1}^{d} \frac{n_i b_i(n)}{(1+|n|)^\beta} + \sum_{i=1}^{d} \frac{n_i \mathbb{1}_{n_i\neq 1} d_i(n)}{|n|^\beta} - \sum_{i=1}^{d} \mathbb{1}_{n_i=1} d_i(n) \sum_{k=|n|+1}^{+\infty} \frac{1}{k^\beta} \\ &\geq \frac{1}{|n|^\beta}\left[\underline{d}(|n|) - \frac{\bar{d}(|n|)}{\beta - 1}\right], \end{aligned}$$

where we used the fact that

$$\sum_{k=x+1}^{+\infty} \frac{1}{k^\beta} \leq \int_x^{+\infty} \frac{dy}{y^\beta} = \frac{1}{(\beta-1)x^{\beta-1}}.$$

Hence it follows from Assumption (3.1) that there exists $\beta > 1$ large enough such that $L\varphi(n) \geq 0$ for all $|n|$ large enough. This entails Condition (a).

We fix such a value of β. Using that

$$\sup_{n\in\mathbb{N}^d} V(n) = \sum_{k=1}^{+\infty} \frac{1}{k^\alpha} \leq 1 + \int_1^\infty \frac{dx}{x^\alpha} = \frac{\alpha}{\alpha-1}$$

and

$$\varphi(n) \geq \int_{|n|+1}^\infty \frac{dx}{x^\beta} = \frac{(1+|n|)^{1-\beta}}{\beta-1} \geq \frac{|n|^{1-\beta}}{2(\beta-1)}$$

for $|n|$ large enough, we compute for such n

$$\begin{aligned} LV(n) + \frac{V^{1+\varepsilon}(n)}{\varphi^\varepsilon(n)} &\leq \sum_{i=1}^d \frac{n_i b_i(n)}{(|n|+1)^\alpha} - \sum_{i=1}^d \frac{n_i d_i(n) \mathbb{1}_{n_i\neq 1}}{|n|^\alpha} + C|n|^{\varepsilon(\beta-1)} \\ &\leq -\frac{\underline{d}(|n|)}{|n|^\alpha} + C|n|^{\varepsilon(\beta-1)}, \end{aligned}$$

where $C = [\alpha/(\alpha-1)]^{1+\varepsilon}[2(\beta-1)]^\varepsilon$. Choosing $\alpha = 1+\eta/2$ and $\varepsilon = \eta/[2(\beta-1)]$, Assumption (3.2) implies that $LV(n) + \frac{V^{1+\varepsilon}(n)}{\varphi^\varepsilon(n)} \leq 0$ for $n \notin O_m$ with m large enough. Since $\inf_{n\in O_m} \varphi(n) > 0$, we have proved Condition (b). □

4 Application to multidimensional Feller diffusions absorbed when one of the coordinates hits 0

We consider a general multitype Feller diffusion $(X_t, t \geq 0)$ in $\mathbb{R}_+^d$, solution to the stochastic differential equation

$$dX_t^i = \sqrt{\gamma_i X_t^i} dB_t^i + X_t^i r_i(X_t) dt, \quad 1 \leq i \leq d, \tag{4.1}$$

where $(B_t^i, t \geq 0)$ are independent standard Brownian motions, γ_i are positive constants and r_i are measurable maps from $\mathbb{R}_+^d$ to $\mathbb{R}$. From the biological point of view, $r_i(x)$ represents the growth rate per individual of species i in a population of size vector $x \in \mathbb{R}_+^d$. We shall make the following assumption.

Assumption 6. *Assume that, for all $i \in \{1,\dots,d\}$, r_i is locally Hölder on $\mathbb{R}_+^d$ and that there exist $a > 0$ and $0 < \eta < 1$ such that*

$$r_i(x) \leq a^\eta - x_i^\eta, \tag{4.2}$$

and there exist constants $B_a > a$, $C_a > 0$ and $D_a > 0$ such that

$$\sum_{i=1}^d \mathbb{1}_{x_i \geq B_a} r_i(x) \leq C_a \left(\sum_{i=1}^d \mathbb{1}_{x_i \leq a} r_i(x) + D_a \right), \quad \forall x \in \mathbb{R}_+^d. \tag{4.3}$$

Assumptions (4.2) and (4.3) correspond to some form of competitivity for the system of interacting Feller diffusions. Moreover, Assumption (4.2) entail that each coordinate of the process can be upper bounded by the (strong) solution of the one-dimensional Feller diffusion

$$d\hat{X}_t^i = \sqrt{\gamma_i \hat{X}_t^i} dB_t^i + a^\eta \hat{X}_t^i dt, \quad 1 \leq i \leq d, \tag{4.4}$$

with initial value $\hat{X}_0^i = X_0^i$. To prove this, one can apply the transformation $Y_t^i = 2\sqrt{X_t^i/\gamma_i}$ to obtain a SDE of the form

$$dY_t^i = dB_t^i + \hat{r}_i(Y_t)dt, \quad 1 \leq i \leq d,$$

with $\hat{r}_i$ measurable, locally Hölder, hence locally bounded in $(0,+\infty)^d$, and (possibly) singular at $\partial := \mathbb{R}_+^d \setminus (0,+\infty)^d$, and apply classical comparison results to solutions of SDEs with constant diffusion coefficient. Since the SDE (4.4) is non-explosive ($+\infty$ is a natural boundary and 0 is an exit boundary for this diffusion by classical criteria, cf. e.g. [15]), the SDE (4.1) is non explosive. Similarly, Assumption (4.2) implies that $X_t^i \leq \bar{X}_t^i$ where

$$d\bar{X}_t^i = \sqrt{\gamma_i \bar{X}_t^i} dB_t^i + \bar{X}_t^i \left[a^\eta - \left(\bar{X}_t^i\right)^\eta \right] dt, \quad 1 \leq i \leq d, \tag{4.5}$$

for some constant B and with initial value $\bar{X}_0^i = X_0^i$. Since $\bar{X}^i$ is a diffusion on $\mathbb{R}_+$ for which $+\infty$ is an entrance boundary and 0 an exit boundary, we deduce that $\tau_\partial < +\infty$ almost surely.

We also deduce from the fact that r_i is locally bounded and from (4.2) that there is strong existence and pathwise uniqueness for (4.1). The ideas to prove this seem quite standard, but we couldn't find a proof of this fact in our situation, so we explain the main steps below. Assume that $X_0^i > 0$ for all $1 \leq i \leq d$ (otherwise, we will see below what to do). Since the drifts $\hat{r}_i$ of Y^i are locally bounded in $(0,+\infty)^d$, we can use the well-known result of Veretennikov [25] to prove local strong existence and pathwise uniqueness until possible explosion or the first exit time τ_{exit} from all compact subsets of $(0,+\infty)^d$. As explained above, the process is non-explosive. In addition, on the event $\{\tau_{\text{exit}} < \infty\}$, the process admits a left limit at the time τ_{exit} since its drift and diffusion coefficients are bounded on each compact subset of $\mathbb{R}_+^d$. Hence, we can define $X_{\tau_{\text{exit}}}$ as the right limit of X_t at this time, which has at least one zero coordinate, i.e. $\tau_\partial = \tau_{\text{exit}}$. We then set $X_t^j = 0$ for all $t \geq \tau_\partial$ and all j such that $X^j_{\tau_\partial} = 0$. We can then proceed as above for the set of coordinates such that $X^i_{\tau_\partial} > 0$. This gives the existence of a global strong solution.

For pathwise uniqueness, we know that it holds locally between the times of absorption of a coordinate, but only for the process formed by the nonzero coordinates, so we need to prove that no solution of (4.1) starting with one zero coordinate, say $X_0^i = 0$, can have $X_t^i > 0$ for some positive time t. This can be proved using again the comparison between X^i and the Feller diffusions (4.4), since 0 is absorbing (since it is an exit boundary) for this diffusion.

Strong existence and pathwise uniqueness imply well-posedness of the martingale problem, hence the strong Markov property hold on the canonical space with respect to the natural filtration (see e.g. [23]). Since the paths of X are continuous, the hitting times of closed subsets of $\mathbb{R}_+^d$ are stopping times for this filtration. Hence, defining $O_n = \{x \in (0,+\infty)^d : 1/(2n) < x_i < 2n,\ \forall 1 \leq i \leq d\}$ and $U_n = \{x \in (0,+\infty)^d : 1/(2n-1) \leq x_i \leq 2n-1,\ \forall 1 \leq i \leq d\}$ for all $n \geq 1$, it is then clear that the process X satisfies the conditions of Section 2.1 and, in view of the previous construction of the process, it is regularly absorbed. In addition, it follows from Itô's formula and the local boundedness of the coefficients of the SDE that any measurable function $f : \mathbb{R}_+^d \to \mathbb{R}$ twice continuously differentiable on $(0,+\infty)$ belongs to the domain of the weakened generator of X and

$$Lf(x) = \sum_{i=1}^d \frac{\gamma_i x_i b_i(x)}{2} \frac{\partial f}{\partial x_i}(x) + \sum_{i=1}^d x_i \frac{\partial^2 f}{\partial x_i^2}(x), \quad \forall x \in (0,+\infty)^d.$$

We can now state the main result of the section.

Theorem 4.1. *Under Assumption 6, the multi-dimensional Feller diffusion process $(X_t, t \geq 0)$ absorbed when one of its coordinates hits 0 admits a unique quasi-stationnary distribution ν_{QSD} and there exist constants $C, \gamma > 0$ such that, for all probability measure μ on $\mathbb{N}^d$,*

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq C e^{-\gamma t}, \quad \forall t \geq 0.$$

It is straightforward to check that Assumption 6 is satisfied in the competitive Lotka-Volterra case, that is when $r_i(x) = r_i - \sum_{i=1}^d c_{ij}x_j$ with $c_{ij} \geq 0$ and $c_{ii} > 0$ for all $1 \leq i, j \leq d$. Hence Theorem 1.2 is an immediate corollary of Theorem 4.1. Assumption 6 allows for other biologically relevant models. For instance, one can consider ecosystems where the competition among individuals only begins when the population size reaches a level $K > 0$, which leads for instance to the SDE

$$dX_t^i = \sqrt{\gamma_i X_t^i}\, dB_t^i + X_t^i \left(r - \sum_{j=1}^d c_{ij} \, (X_t^j - K)_+ \right) dt, \;\; X_0^i > 0,$$

where c_{ij} are non-negative constants and $c_{ii} > 0$ for all $i, j \in \{1, \dots, d\}$. Note also that a similar approach (*i.e.* using Theorem 2.4 with Lyapunov functions of the form $\prod_{i=1}^d h(x_i)$) can also be used to handle diffusion processes evolving in bounded boxes.

Proof. Up to a linear scaling of the coordinates, we can assume without loss of generality that $\gamma_i = 2$ for all $1 \leq i \leq d$, so we will only consider this case from now on. Note that Assumption 6 is not modified by the rescaling.

We shall make use of Corollary 2.8. We divide the proof into five steps, respectively devoted to the construction of a function φ satisfying Conditions (a), of a function V satisfying Condition (b), to the proof of (2.9), to the proof of a local Harnack inequality, and to check Assumption 2, Assumption 3 and Assumption 4.

Step 1: construction of a function φ satisfying Condition (a).

Recall the definition of the constants $a > 0$ and $B_a > a$ from Assumption 6. We use the following lemma, proved at the end of this section.

Lemma 4.2. *There exists $M > 0$ such that, for all $\beta \geq M$, there exists a function $h_\beta : \mathbb{R}_+ \to \mathbb{R}_+$ twice continuously differentiable on $(0, +\infty)$ such that*

$$h_\beta(x) = \begin{cases} 4x^2/a^2 & \text{if } x \in [0, a/2], \\ B_a^\beta (2x)^{-\beta} & \text{if } x \geq B_a, \end{cases}$$

$h_\beta(x) \geq 1$ for all $x \in [a/2, a]$, h_β is nonincreasing and convex on $[a, +\infty)$,

$$M' := \sup_{\beta \geq M} \sup_{x \in [a/2, a]} |h'_\beta(x)| < +\infty \quad \text{and} \quad M'' := \sup_{\beta \geq M} \sup_{x \in [a/2, a]} |h''_\beta(x)| < +\infty.$$

We set $\beta = M + (2 \vee aM')/C_a + 1$,

$$\varphi(x) = \prod_{i=1}^d h_\beta(x_i), \quad \forall x \in \mathbb{R}_+^d$$

and we shall prove that φ satisfies Condition (a). We have

$$\frac{L\varphi(x)}{\varphi(x)} = \sum_{i=1}^{d} \frac{x_i h'_\beta(x_i) r_i(x) + x_i h''_\beta(x)}{h_\beta(x_i)}.$$

Now, it follows from the properties of h_β and Assumptions 6 that, for all $x \in \mathbb{R}_+$ and all $1 \leq i \leq d$,

$$\frac{x_i h'_\beta(x_i) r_i(x) + x_i h''_\beta(x)}{h_\beta(x_i)} \geq \begin{cases} 0 & \text{if } x_i \geq a, \\ -\beta r_i(x) & \text{if } x_i \geq B_a, \\ 2r_i(x) + \frac{2}{x_i} & \text{if } x_i \leq a/2, \\ aM'(r_i(x) - a^\eta) - a^{1+\eta}M' - aM'' & \text{if } a/2 \leq x_i \leq a, \end{cases}$$

where we used in the last inequality the fact that $r_i(x) - a^\eta \leq 0$ for all x. Using once again this property, we deduce that, for some constant B independent of $\beta \geq M$,

$$\frac{x_i h'_\beta(x_i) r_i(x) + x_i h''_\beta(x)}{h_\beta(x_i)} \geq \begin{cases} 0 & \text{if } x_i \geq a, \\ -\beta r_i(x) & \text{if } x_i \geq B_a, \\ (2 \vee aM')r_i(x) + \frac{2}{x_i} - B & \text{if } x_i \leq a. \end{cases} \quad (4.6)$$

Hence, for all $x \in \mathbb{R}^d_+$,

$$\frac{L\varphi(x)}{\varphi(x)} \geq -\beta \sum_{i=1}^{d} \mathbb{1}_{x_i \geq B_a} r_i(x) + \sum_{i=1}^{d} \mathbb{1}_{x_i \leq a} \left((2 \vee aM')r_i(x) + \frac{2}{x_i} \right) - dB.$$

This and Assumption (4.3) imply that

$$\begin{aligned} \frac{L\varphi(x)}{\varphi(x)} &\geq -\sum_{i=1}^{d} \mathbb{1}_{x_i \geq B_a} r_i(x) + 2\sum_{i=1}^{d} \mathbb{1}_{x_i \leq a} \frac{1}{x_i} - dB - (2 \vee aM')D_a \\ &\geq \sum_{i=1}^{d} \left(x_i^\eta + \frac{2}{x_i} \right) - B', \end{aligned}$$

for some constant B', where we used Assumption (4.2) in the last inequality.

Hence, there exists $n \geq 1$ and a constant $C > 0$ such that

$$L\varphi(x) \geq -C\mathbb{1}_{x \in O_n}.$$

This ends the proof that φ satisfies Condition (a).

Step 2: construction of a function V satisfying Condition (b) and verification that (V,φ) is a couple of admissible functions.

For V, we define

$$V(x)=\prod_{i=1}^{d} g(x_i),\quad \forall x\in\mathbb{R}_+^d,$$

where the function $g:\mathbb{R}_+\to\mathbb{R}_+$ is twice continuously differentiable on $(0,+\infty)$, increasing concave and such that

$$g(x)=\begin{cases} x^{\gamma} & \text{if } x\leq 1\\ \delta-x^{-\eta/2} & \text{if } x\geq 2,\end{cases}$$

for some constants $\gamma<1$ and $\delta>0$ and where η is defined in Assumption (4.2). Since $g'(1)=\gamma$ and $g'(2)=\eta 2^{-2-\eta/2}$, it is possible to find $\delta>0$ such that such a function g exists as soon as $\eta 2^{-2-\eta/2}<\gamma$. Hence, we shall assume that γ belongs to the non-empty interval $(\eta 2^{-2-\eta/2},1)$. We have

$$\frac{LV(x)}{V(x)}=\sum_{i=1}^{d}\frac{x_i g'(x_i)r_i(x)+x_i g''(x_i)}{g(x_i)}$$

and

$$\frac{x_i g'(x_i)r_i(x)+x_i g''(x_i)}{g(x_i)}\leq\begin{cases}\gamma r_i(x)-\dfrac{\gamma(1-\gamma)}{x_i} & \text{if } x_i\leq 1,\\ 2a^{\eta}\sup\limits_{1\leq x\leq 2} g'(x) & \text{if } 1\leq x_i\leq 2,\\ \dfrac{\eta r_i(x)x_i^{-\eta/2}}{2(\delta-x_i^{-\eta/2})} & \text{if } x_i\geq 2.\end{cases}$$

We deduce from Assumptions (4.2) that there exist constants $B',B''>0$ such that

$$\frac{x_i g'(x_i)r_i(x)+x_i g''(x)}{g(x_i)}\leq B'-\begin{cases}\frac{\gamma(1-\gamma)}{x_i} & \text{if } x_i\leq 1,\\ 0 & \text{if } 1\leq x_i\leq 2,\\ B''x_i^{\eta/2} & \text{if } x_i\geq 2.\end{cases}\tag{4.7}$$

Thus, since $h_\beta(x_i)\geq D_\beta\left(x_i^2\wedge x_i^{-\beta}\right)$ for some constant $D_\beta>0$ and since

$g(x_i) \leq \delta$,

$$\frac{LV(x)}{V(x)} + \frac{V(x)^\varepsilon}{\varphi(x)^\varepsilon} \leq B'd - \gamma(1-\gamma)\sum_{i=1}^d \frac{\mathbb{1}_{x_i \leq 1}}{x_i} - B''\sum_{i=1}^d \mathbb{1}_{x_i \geq 2} x_i^{\eta/2}$$
$$+ \left(\frac{\delta}{D_\beta}\right)^{d\varepsilon} \prod_{i=1}^d \left(x_i^{\varepsilon\beta} \vee x_i^{-2\varepsilon}\right)$$
$$\leq B'd + \gamma(1-\gamma) + B''2^{\eta/2} - \gamma(1-\gamma)\left(\inf_i x_i\right)^{-1} - B''\left(\sup_i x_i\right)^{\eta/2}$$
$$+ \left(\frac{\delta}{D_\beta}\right)^{d\varepsilon}\left[\left(\sup_i x_i\right)^{\beta d\varepsilon} + \left(\inf_i x_i\right)^{-2d\varepsilon}\right].$$

Therefore, choosing $\varepsilon > 0$ such that $d(1+\alpha)\varepsilon < 1$ and $\beta d\varepsilon < \eta/2$, $LV(x) + V(x)^{1+\varepsilon}/\varphi(x)^\varepsilon \leq 0$ for all $x \in \mathbb{R}_+^d$ such that $\inf_i x_i$ is small enough or $\sup_i x_i$ is big enough. Since LV is bounded from above by (4.7) and since V and φ are positive continuous on any compact subset of $(0, +\infty)^d$, we have proved Condition (b).

We can now check that (V, φ) is a couple of admissible functions. First, V and φ are both bounded, positive on $(0, +\infty)$ and vanishing on ∂. They both belong to the domain of the weakened infinitesimal generator of X. Since the function g/h_β is positive continuous on $(0, +\infty)$ and

$$\frac{g(x)}{h_\beta(x)} = \begin{cases} x^{\gamma-2} & \text{if } x \leq 1 \wedge a/2, \\ (\delta - x^{-\eta/2})(2x)^\beta / B_a^\beta & \text{if } x \geq B_a \vee 2, \end{cases}$$

we deduce that $\inf_{x\in(0,+\infty)} g(x)/h_\beta(x) > 0$, hence (2.3) holds true, and (2.4) is also clear. Since V is continuous on $\mathbb{R}_+^d$ and $X_{T_n} \to X_{\tau_\partial}$ almost surely, (2.5) also holds true. Finally, since LV is bounded from above and $L\varphi$ is bounded from below, we have proved that (V, φ) is a couple of admissible functions.

Step 3: proof of (2.9).

Using the upper bound $X_t^i \leq \bar{X}_t^i$ for all $t \geq 0$ and $1 \leq i \leq d$, where $\bar{X}^i$ is solution to the SDE (4.5), and noting that the processes $(\bar{X}^i)_{1\leq i\leq d}$ are independent, we have for all $x \in \mathbb{R}_+^d$ and all $t_2 > 0$,

$$\mathbb{P}_x(t_2 < \tau_\partial) \leq \prod_{i=1}^d \mathbb{P}_{x_i}(\bar{X}_{t_2}^i > 0).$$

Now, there exist constants D and D' such that

$$\mathbb{P}_{x_i}(\bar{X}_{t_2}^i > 0) \leq (Dx_i) \wedge 1 \leq D' g(x_i) \quad \text{for all } x_i > 0. \tag{4.8}$$

To prove this, we can consider a scale function s of the diffusion $\bar{X}^i$ such that $s(0)=0$. Using the expression of the scale function and the speed measure (see e.g. [23, V.52]), one easily checks that $s(x_i)\sim x_i$ when $x_i\to 0$ and that Proposition 4.9 of [8] is satisfied, so that $\mathbb{P}_{x_i}(\bar{X}^i_{t_2}>0)\leq Ms(x_i)$ for some $M>0$. Since $s(x_i)\sim x_i$ when $x_i\to 0$, (4.8) is proved and hence (2.9) holds true.

Step 4: Harnack inequality for u. Consider a bounded measurable function f and define the application $u:(t,x)\in\mathbb{R}_+\times E\mapsto \mathbb{E}_x(f(X_t))$. Our aim is to prove that, for all $m\geq 1$, there exist two constants $N_m>0$ and $\delta_m>0$, which do not depend on f, such that

$$u(\delta_m+\delta_m^2,x)\leq N_m u(\delta_m+2\delta_m^2,y),\ \text{ for all } x,y\in O_m \text{ such that } |x-y|\leq\delta_m. \tag{4.9}$$

First, the function u is continuous on $\mathbb{R}_+^*\times E$. Indeed, since the SDE (4.1) has Hölder coefficients in O_m, using the same approach as in the proof of [24, Thm 7.2.4], one deduces that, for all $x\in E$ and for all $\varepsilon>0$, there exists $\delta>0$ such that, for all $y\in B(x,\delta)$, all $s,t\in[\delta,2\delta]$ and all bounded measurable function $\varphi:E\to\mathbb{R}$,

$$\left|\mathbb{E}_x\left(\varphi(X_s)\right)-\mathbb{E}_y\left(\varphi(X_t)\right)\right|\leq\varepsilon\|\varphi\|_\infty.$$

This result applied to the function $\varphi:x\mapsto\mathbb{E}_x\left(f(X_{t-\delta})\right)$ for $t\geq\delta$ and the Markov property entails the continuity of u on $\mathbb{R}_+^*\times E$.

Second, fix $m\geq 1$ and let K be a compact set with C^∞ boundary such that $O_m\subset K\subset E$ and such that $d(O_m,\partial K)>0$. We set $\delta=d(O_m,\partial K)/3$. Let $(h_n)_{n\geq 1}$ be a sequence of bounded C^∞ functions from $([\delta,\infty[\times\partial K)\cup(\{\delta\}\times K)$ to $\mathbb{R}$ such that $(h_n)_{n\geq 1}$ converges uniformly locally toward u restricted to $([\delta,\infty[\times\partial K)\cup(\{\delta\}\times K)$ (such a sequence exists because of the continuity of u). For all $n\geq 1$, we let $u_n:[\delta,+\infty[\times K\to\mathbb{R}$ be the solution to the linear parabolic equation

$$\begin{cases}\partial_t u_n=\displaystyle\sum_{i=1}^d x_i\frac{\partial^2 u_n}{\partial x_i^2}+x_i r_i(x)\frac{\partial u_n}{\partial x_i},\\ u_n(t,x)=h_n(t,x),\ \forall (t,x)\in([\delta,\infty[\times\partial K)\cup(\{\delta\}\times K).\end{cases}$$

By [18, Thm 5.1.15], for all $n\geq 1$, u_n is of regularity $C^{1,2}$. Moreover, using the Harnack inequality provided by [16, Theorem 1.1] (with $\theta=2$ and $R=\delta$), we deduce that there exists a constant $N>0$ which does not depend on f nor on n such that

$$u_n(\delta+\delta^2,x)\leq N u_n(\delta+2\delta^2,y),\ \text{ for all } x,y\in O_m \text{ such that } |x-y|\leq\delta. \tag{4.10}$$

In particular, applying Itô's formula to $s\mapsto u_n(\delta+t-s,X_s)$ at time $\tau_K\wedge t$ and taking the expectation, one deduces that, for all $t\geq 0$,

$$\begin{aligned} u_n(\delta+t,x) &= \mathbb{E}_x\left[u_n(\delta+t-\tau_K\wedge t, X_{\tau_K\wedge t})\right] \\ &= \mathbb{E}_x\left(\mathbb{1}_{\tau_K\leq t}u_n(t+\delta-\tau_K,X_{\tau_K})\right)+\mathbb{E}_x\left(\mathbb{1}_{\tau_K>t}u_n(\delta,X_t)\right) \\ &= \mathbb{E}_x\left(\mathbb{1}_{\tau_K\leq t}h_n(t+\delta-\tau_K,X_{\tau_K})\right)+\mathbb{E}_x\left(\mathbb{1}_{\tau_K>t}h_n(\delta,X_t)\right). \end{aligned}$$

By Lebesgue's theorem, the last quantity converges when $n\to+\infty$ to

$$\begin{aligned} &\mathbb{E}_x\left(\mathbb{1}_{\tau_K\leq t}u(t+\delta-\tau_K,X_{\tau_K})\right)+\mathbb{E}_x\left(\mathbb{1}_{\tau_K>t}u(\delta,X_t)\right) \\ &\qquad = \mathbb{E}_x\left(\mathbb{1}_{\tau_K\leq t}\mathbb{E}_{X_{\tau_K}}(f(X_{t+\delta-\tau_K})\right)+\mathbb{E}_x\left(\mathbb{1}_{\tau_K>t}\mathbb{E}_{X_t}(f(X_\delta))\right) \\ &\qquad = \mathbb{E}_x(f(X_{\delta+t}))=u(\delta+t,x), \end{aligned}$$

where we used the strong Markov property at time τ_K.

We deduce from this convergence and the Harnack inequalities (4.10) that there exist two constants $N_m>0$ and $\delta_m>0$, which do not depend on f, such that (4.9) holds true.

Step 5 : proof that Assumptions 2, 3 and 4 are satisfied. Fix $x_1\in O_1$ and let ν denote the conditional law $\mathbb{P}_{x_1}(X_{\delta_1+\delta_1^2}\in\cdot\mid\delta_1+\delta_1^2<\tau_\partial)$. Then the Harnack inequality (4.9) entails that, for all $x\in O_1$ such that $|x-x_1|\leq\delta_1$,

$$\mathbb{P}_x(X_{\delta_1+2\delta_1^2}\in\cdot)\geq\frac{\mathbb{P}_{x_1}(\delta_1+\delta_1^2<\tau_\partial)}{N_1}\,\nu.$$

Replacing O_1 by its intersection with the open ball with center x_1 and radius δ_1, this implies that Assumption 2 is satisfied for $k_0=1$ (the second part of this assumption is a classical property for locally elliptic diffusion processes).

Assumption 3 is a direct consequence of the domination by solutions to (4.5), since these solutions come down from infinity and hit 0 in finite time almost surely (cf. e.g. [2]).

Finally, Assumption 4 is a consequence of (4.9). Indeed, for any fixed m and for all $t\geq\delta_m+2\delta_m^2$, this equation applied to $f(x)=\mathbb{P}_x(t-\delta_m-2\delta_m^2<\tau_\partial)$ and the Markov property entails that

$$\mathbb{P}_x(t-\delta_m^2<\tau_\partial)\leq N_m\mathbb{P}_y(t<\tau_\partial),\ \text{ for all } x,y\in O_m \text{ such that } |x-y|\leq\delta_m.$$

Since $s \mapsto \mathbb{P}_x(s < \tau_\partial)$ is non-increasing, we deduce that

$$\mathbb{P}_x(t < \tau_\partial) \leq N_m \mathbb{P}_y(t < \tau_\partial), \text{ for all } x, y \in O_m \text{ such that } |x - y| \leq \delta_m.$$

Since O_m has a finite diameter and is connected, we deduce that there exists N'_m such that, for all $t \geq \delta_m + 2\delta_m^2$,

$$\mathbb{P}_x(t < \tau_\partial) \leq N'_m \mathbb{P}_y(t < \tau_\partial), \text{ for all } x, y \in O_m.$$

Now, for $t \leq \delta_m + 2\delta_m^2$, we simply use the fact that $x \mapsto \mathbb{P}_x(\delta_m + 2\delta_m^2 < \tau_\partial)$ is uniformly bounded from below on O_m by a constant $1/N''_m > 0$. In particular,

$$\mathbb{P}_x(t < \tau_\partial) \leq 1 \leq N''_m \mathbb{P}_y(\delta_m + 2\delta_m^2 < \tau_\partial) \leq N''_m \mathbb{P}_y(t < \tau_\partial), \text{ for all } x, y \in O_m.$$

As a consequence, Assumption 4 is satisfied.

Finally, we deduce from Steps 1, 2, 3 and 5 that all the assumptions of Theorem 2.8 are satisfied. This concludes the proof of Theorem 1.2. □

Proof of Lemma 4.2. We consider a C^∞ function $\varphi : \mathbb{R} \to [0, 1]$ such that $\varphi(x) = 0$ for all $x \leq a/2$ and $\varphi(x) = 1$ for all $x \geq a$. For any $\beta > 0$, we define the function h_β as

$$h_\beta(x) = \begin{cases} 4x^2/a^2 & \text{if } x \in [0, a/2], \\ P_1(x) & \text{if } x \in [a/2, a], \\ P_2(x) & \text{if } x \in [a, B_a], \\ B_a^\beta (2x)^{-\beta} & \text{if } x \geq B_a, \end{cases}$$

where

$$P_1(x) = 4x^2/a^2(1 - \varphi(x)) + \varphi(x) P_2(x)$$

and

$$P_2(x) = 2^{-\beta} - \frac{\beta 2^{-\beta}}{B_a}(x - B_a) + \frac{\beta(\beta + 1)2^{-\beta-1}}{B_a^2}(x - B_a)^2 + C_\beta (x - B_a)^4,$$

with

$$C_\beta = \frac{-2^{-\beta} + \beta 2^{-\beta}(a/B_a - 1) - \beta(\beta + 1)2^{-\beta-1}(a/B_a - 1)^2 + 1}{(a - B_a)^4}.$$

Note that the coefficients of the polynomial $P-2$ have been chosen so that h_β is C^2 at B_a and $P_2(a)=1$.

There exists $M>0$ such that $C_\beta>0$ for all $\beta\geq M$, so that the first and second derivative of P_2 satisfy, for all $\beta\geq M$,

$$P_2'(x)=-\frac{\beta 2^{-\beta}}{B_a}+\frac{\beta(\beta+1)2^{-\beta}}{B_a^2}(x-B_a)+4C_\beta(x-B_a)^3\leq 0,\ \forall x\leq B_a,$$
$$P_2''(x)=\frac{\beta(\beta+1)2^{-\beta}}{B_a^2}+12C_\beta(x-B_a)^2\geq 0,\ \forall x\in\mathbb{R}.$$

In particular, P_2 is decreasing and convex on $(-\infty,B_a]$. Moreover, since $P_2(a)=1$, we deduce that $P_2(x)\geq 1$ for all $x\leq a$.

Finally, one easily deduces from the above definitions and properties that, for all $\beta\geq M$, h_β is of regularity C^2, that $h_\beta(x)\geq 1$ for all $x\in[a/2,a]$, that h_β is decreasing and convex on $[a,+\infty)$. The finiteness of M' and M'' are simple consequences of the fact that the coefficients of P_2 are uniformly bounded in $\beta\geq M$. This concludes the proof of Lemma 4.2. □

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