On the link between infinite horizon control and quasi-stationary distributions

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Abstract

We study infinite horizon control of continuous-time non-linear branching processes with almost sure extinction for general (positive or negative) discount. Our main goal is to study the link between infinite horizon control of these processes and an optimization problem involving their quasi-stationary distributions and the corresponding extinction rates. More precisely, we obtain an equivalent of the value function when the discount parameter is close to the threshold where the value function becomes infinite, and we characterize the optimal Markov control in this limit. To achieve this, we present a new proof of the dynamic programming principle based upon a pseudo-Markov property for controlled jump processes. We also prove the convergence to a unique quasi-stationary distribution of non-linear branching processes controlled by a Markov control conditioned on non-extinction.

Keywords. Optimal stochastic control, infinite horizon control, branching process, quasi-stationary distribution, dynamic programming.

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1   Introduction

Given a family of controlled stochastic processes $(X_t^{x,\alpha}, t \geq 0)$ in some measurable state space $(S, \mathcal{S})$, adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$, where $x \in S$ is the initial value of $X^{x,\alpha}$, $\alpha = (\alpha_s, s \geq 0)$ is a $(\mathcal{F}_t)_{t \geq 0}$-predictable process belonging to a given set $\mathcal{A}$ of admissible controls, the classical infinite horizon control problem is formulated as follows: given $\beta < 0$ and a bounded measurable function f, one defines the value function as

$$v_\beta(x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{\beta t} f(X_t^{x,\alpha}, \alpha_t)\, dt \right]. \qquad (1)$$

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The infinite horizon control problem consists in characterizing the optimal control α^* realizing the above supremum (when it exists).

In the case where $(X_t^{x,\alpha}, t \geq 0)$ is an ergodic Markov process for all Markov control α, i.e. all control of the form $\alpha_s = a(X_{s-}^{x,\alpha})$ for some measurable function a, the infinite horizon control problem when $\beta \uparrow 0$ can be formulated as an optimization problem on (ergodic) invariant distributions. More precisely, If we call $\mathcal{A}_M$ the set of Markov controls, using the ergodic theorem, one expects that, when $\beta \uparrow 0$,

$$v_\beta(x) \sim \frac{1}{\beta} \sup_{\alpha \in \mathcal{A}_M} \mu^\alpha(f^\alpha), \qquad (2)$$

where, for all $\alpha \in \mathcal{A}_M$, $f^\alpha : x \mapsto f(x, a(x))$ and μ^α is the invariant distribution of $X^{x,\alpha}$ for all $x \in S$. For further details, we refer the reader to, e.g., [Put94] for discrete-time Markov chains, [GHL09] for the continuous-time case and [ABG12] for diffusion processes.

If one assumes that for all $\alpha \in \mathcal{A}$ and all $x \in S$, the process $X^{x,\alpha}$ is a.s. absorbed after a finite time at some point ∂, and if one assumes $f(\partial) = 0$ and $f(x) > 0$ for all $x \in S \setminus \{\partial\}$, then one expects that the ergodic control problem (1) favors controls α for which the absorption time of $X^{x,\alpha}$ is longer. The relation (2) clearly does not apply in this case since $\mu_\alpha = \delta_\partial$ and $f(\partial) = 0$. This suggests that this new problem needs to be studied using quasi-stationary distributions (QSD for short) instead of the stationary distribution δ_∂.

For a Markov process $(Y_t, t \geq 0)$ taking values in $S \cup \{\partial\}$ a.s. absorbed in finite time at ∂, a probability measure π on S is said to be quasi-stationary if

$$\mathbb{P}_\pi(Y_t \in \cdot \mid t < \tau) = \pi, \qquad \forall t \geq 0,$$

where $\tau := \inf\{t \geq 0 : Y_t = \partial\}$ is the absorption time of Y and $\mathbb{P}_\pi$ is the law of Y given that $Y_0 \sim \pi$. It is well-known [MV12] that, if π is a QSD, then the absorption time starting from π is exponentially distributed with some parameter $\lambda > 0$ called the absorption rate of the QSD π:

$$\mathbb{P}_\pi(t < \tau) = e^{-\lambda t}, \quad \forall t \geq 0.$$

Recently, new criteria to prove the existence, uniqueness and exponential convergence of conditional distribution of general absorbed Markov processes were obtained in [CV15a].

The goal of this article is to make the link between infinite horizon control of absorbed processes and quasi-stationary distributions, by proving an extension of (2) in the case where $f(\partial) = 0$.

We will restrict in this work to continuous-time controlled processes in $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ which are *non-linear branching processes*. We call them *branching processes* since they allow simultaneous births from a single individual, as in the continuous-time counterpart of Galton-Watson processes, and deaths only occur singly. In addition, the process is absorbed at the state $\partial = 0$, when the population goes extinct. However, we consider more general processes than processes satisfying the branching property (processes with linear birth and death rates), which explains the term *non-linear*. In such a general settings, very few results on QSD are known, except for processes satisfying the branching property [AN72], and for state-dependent branching processes in discrete time [Gos01], but the last reference does not obtain uniform convergence of conditional distributions, which is needed for our analysis. A particular

case is given by classical birth and death processes, for which quasi-stationary properties have been studied for a long time [Cav78, vD91, FMP92]. More recently, the topic of QSD for birth and death processes has been further studied in [MSV13, CV15a, CV15b] and [CCM14]. The first three of the last references study the exponential convergence in total variation of conditional distributions to a unique QSD, uniformly w.r.t. the initial distribution. The new results we obtain here on QSD in the setting of non-linear branching processes are exactly of this type.

Non-linear branching processes can be controlled in the sense that the birth and death rates and the progeny distribution may depend on a control parameter. Note that a controlled process may not satisfy the branching property even if each process with a constant control satisfies it. Hence, for the sake of simplicity, we will call our processes *controlled branching processes* and omit the *non-linear* which is implicit.

The optimal control of stochastic or deterministic population dynamics like non-linear branching processes is an important topic in various biological domains. We can cite for example conservation biology, where the controller tries to favor the survival of an endangered species [HM99, MHC01] , or medecine and agronomy, where the controller wants to favor the extinction of a population of pathogens in some disease or of pests in an agricultural process [SP00]. Other examples of control problems in fishery, agronomy, bio-reactors or tumor growth may be found e.g. in [LW07, AAC11].

Section 2 is devoted to the definition of the controlled non-linear branching processes and to the proof of preliminary properties. Using the criteria of [CV15a], we also state in Section 2 and prove in Section 5 that for all Markov control α, the controlled branching process $X^{x,\alpha}$ admits a unique quasi-stationary distribution π^α with absorption rate $\lambda^\alpha > 0$, and that the conditional distributions converge exponentially and uniformly in total variation to the QSD. We extend in Section 3 the problem of infinite horizon control (1) to positive values of β, and we also state our main results on infinite horizon control, among which the fact that, if $f \geq 0$ and $f(0, \cdot) \equiv 0$, then, when $\beta \to \lambda_* := \inf_{\alpha \in \mathcal{A}_M} \lambda^\alpha$,

$$v_\beta(x) \sim \frac{1}{\lambda_* - \beta} \sup_{\alpha \in \mathcal{A}_M \text{ s.t. } \lambda^\alpha = \lambda_*} \pi^\alpha(f^\alpha)\eta^\alpha(x),$$

where η^α is the eigenvector of the generator of X^α corresponding to the eigenvalue λ^α. This result deals with control problems favoring survival of the population. We also state similar results on control problems in favor of extinction. In Section 4, we prove that the value function is solution to the Hamilton-Jacobi-Bellman equation, and that the optimal control belongs to the class of Markov controls. These results are to a large extent classical, but the proofs we give are original. Our analysis relies on the analogy with the theory of controlled diffusions using a Poisson random measure playing the role of the Brownian motion. Section 5 is devoted to proofs of our results on QSD. Finally, we give the proofs of the main results of Section 3 in Section 6.

2 Controlled continuous-time non-linear branching processes

2.1 Definition and first properties

Let A be the control space that is assumed to be Polish. Consider a population which evolves like a controlled non-linear branching process. More precisely, if $n \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ is the size of the population, then the (global) death rate is given by $d_n : A \to \mathbb{R}_+$, the (global) birth rate by $b_n : A \to \mathbb{R}_+$ and the probability to have $k \in \mathbb{N} := \{1, 2, \ldots\}$ offsprings by $p_{n,k} : A \to [0, 1]$, where

$$\sum_{k=1}^{+\infty} p_{n,k}(a) = 1, \quad \forall\, n \in \mathbb{Z}_+, \, a \in A.$$

Moreover, we suppose that

$$b_0(a) = d_0(a) = 0, \quad \forall\, a \in A,$$

so that $\partial = 0$ is an absorbing state of the population process.

Let Q_1 and Q_2 be two independent Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with Lebesgue intensity measures. Denote by $(\mathcal{F}_s)_{s \geq 0}$ the filtration generated by Q_1 and Q_2, that is,

$$\mathcal{F}_s := \sigma\left(Q_1(B), Q_2(B); \; B \in \mathcal{B}([0, s]) \otimes \mathcal{B}(\mathbb{R}_+)\right).$$

Let $\mathcal{A}$ be the collection of $(\mathcal{F}_s)_{s \geq 0}$-predictable processes valued in A. An element of $\mathcal{A}$ is called an admissible control.

We want to characterize the population $X^{x,\alpha}$ controlled by $\alpha \in \mathcal{A}$ starting from $x \in \mathbb{Z}_+$ as the solution of the following SDE:

$$X_s^{x,\alpha} = x + \int_{(0,s] \times \mathbb{R}_+} \left(\sum_{k \geq 1} k \mathbb{1}_{I_k\left(X_{\theta-}^{x,\alpha}, \alpha_\theta\right)}(z) \right) Q_1(d\theta, dz)$$

$$- \int_{(0,s] \times \mathbb{R}_+} \mathbb{1}_{\left[0, d\left(X_{\theta-}^{x,\alpha}, \alpha_\theta\right)\right)}(z) \, Q_2(d\theta, dz), \quad \forall\, s \geq 0, \mathbb{P} - \text{a.s.}, \quad (3)$$

where $X_{\theta-}^{x,\alpha}$ denotes the left-hand limit of $X^{x,\alpha}$ at time θ and

$$I_{n,k}(a) := \left[b_n(a) \sum_{l=1}^{k-1} p_{n,l}(a), \; b_n(a) \sum_{l=1}^{k} p_{n,l}(a) \right).$$

The proposition below ensures that, under suitable conditions, the process $X^{x,\alpha}$ is well-defined.

Hypothesis 1. *Assume that:*
(i) b_n, d_n and $p_{n,k}$ are measurable;
(ii) there exists $\bar{b} > 0$ such that

$$b_n(a) \leq \bar{b} n, \quad \forall\, n \in \mathbb{Z}_+, \, a \in A;$$

(iii) there exists $(\bar{d}_n)_{n \in \mathbb{Z}_+} \in (\mathbb{R}_+)^{\mathbb{Z}_+}$ such that

$$d_n(a) \leq \bar{d}_n, \quad \forall\, n \in \mathbb{Z}_+, \, a \in A;$$

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(iv) *there exists $M > 0$ such that*

$$\sum_{k=1}^{\infty} k p_{n,k}(a) \leq M, \quad \forall n \in \mathbb{Z}_+, a \in A.$$

Assumption (ii) means that the birth rate per individual is bounded by $\bar{b}$, regardless of the population size and the control. Assumption (iv) means that the number of offspring at each birth event has bounded first moment, regardless of the population size and the control.

Proposition 1. *Under Hypothesis 1, there exists a unique (up to indistinguishability) càdlàg and adapted process solution of* (3). *In addition, it satisfies*

$$\mathbb{E}\left[\sup_{0 \leq \theta \leq s}\{X_\theta^{x,\alpha}\}\right] \leq x \mathrm{e}^{\bar{b}Ms}. \tag{4}$$

Proof. Since the process $X^{x,\alpha}$ is piecewise constant, it can be constructed for each ω by considering recursively the sequence of atoms of the Poisson measures for which the indicator functions in the integrals are non-zero. This proves the existence and uniqueness of the solution of (3), until the first accumulation point of the sequence of jump times. Therefore, in order to ensure existence and uniqueness of a global solution of (3), one needs to prove that this first accumulation point is infinite. Since Hypothesis 1 ensures that the jump rates are bounded as long as the size of the population is, it suffices to prove that $\tau_n \to +\infty$ as $n \to +\infty$, where

$$\tau_n := \inf\{s \geq 0; \ X_s^{x,\alpha} \geq n\}.$$

We shall prove this and (4) simultaneously. Denote $X_s^* := \sup_{0 \leq \theta \leq s}\{X_\theta^{x,\alpha}\}$. One clearly has

$$X_{s \wedge \tau_n}^* \leq x + \int_{(0,s\wedge\tau_n]\times\mathbb{R}_+} \sum_{k\geq1} k \mathbb{1}_{I_k\left(X_{\theta-}^{x,\alpha},\alpha_\theta\right)}(z) \, Q_1(d\theta,dz).$$

It yields

$$\mathbb{E}\left[X_{s\wedge\tau_n}^*\right] \leq x + \mathbb{E}\left[\int_0^{s\wedge\tau_n} b\left(X_\theta^{x,\alpha},\alpha_\theta\right)\sum_{k\geq1} k p_k\left(X_\theta^{x,\alpha},\alpha_\theta\right)d\theta\right]$$

$$\leq x + \bar{b}M\mathbb{E}\left[\int_0^s X_{\theta\wedge\tau_n}^* \, d\theta\right].$$

Using Gronwall's lemma, we obtain

$$\mathbb{E}\left[X_{s\wedge\tau_n}^*\right] \leq x \mathrm{e}^{\bar{b}Ms}.$$

Since the r.h.s. does not depend on n, we deduce that τ_n converges almost surely to infinity. By Fatou's lemma, we conclude that the inequality (4) holds. $\square$

The last result shows the existence and uniqueness of the solution of (3) except on the event (of zero probability) where there is accumulation of jump events in a bounded time interval. We shall call a time of accumulation of jumps

an explosion time. For questions of measurability, we need to define precisely the process $X^{x,\alpha}$ on this event. To this aim, we assume that this process takes values in $\mathbb{Z}_+ \cup \{\infty\}$, and we pose $X_s = \infty$ after the first explosion time. The process defined this way is clearly adapted w.r.t. $(\mathcal{F}_s)_{s\geq 0}$. A more standard definition of $X^{x,\alpha}$, for example constant equal to 0 on the event of explosion, would require to complete the filtration to obtain an adapted process, but this would pose problems to prove the pseudo-Markov property and the dynamic programming principle of Section 4.

2.2 Markov controls and quasi-stationary distributions

In the sequel, we need to consider a subclass of the admissible controls, the so-called Markov controls. For any sequence $(\alpha_n)_{n\in\mathbb{Z}_+}$ valued in A, consider the following SDE:

$$
X_s = x + \int_{(0,s]\times\mathbb{R}_+} \left(\sum_{k\geq 1} k \mathbb{1}_{I_k^\alpha(X_{\theta-})}(z) \right) Q_1(d\theta, dz)
$$
$$
- \int_{(0,s]\times\mathbb{R}_+} \mathbb{1}_{[0,d^\alpha(X_{\theta-}))}(z)\, Q_2(d\theta, dz), \quad \forall s \geq 0, \mathbb{P}-\text{a.s.}, \quad (5)
$$

where $d^\alpha(n) := d_n(\alpha_n)$ and $I_k^\alpha(n) := I_{n,k}(\alpha_n)$ for all $k, n \geq 0$. By the arguments of Proposition 1, there is a unique (up to indistinguishability) càdlàg and adapted process solution to the SDE above. In addition, this is the controlled process, solution to (3) for the (admissible) control $(s,\omega) \mapsto \alpha(X_{s-}(\omega))$, where we use the notation $\alpha(n) = \alpha_n$. As a consequence, by abuse of notation, α will refer in the sequel both to the sequence $(\alpha_n)_{n\in\mathbb{Z}_+}$ and to the corresponding admissible control. Since in this case $X^{x,\alpha}$ is a Markov process, we call α a Markov control. Denote by $\mathcal{A}_M$ the collection of Markov controls.

For all $\alpha \in \mathcal{A}_M$, the process $X^{x,\alpha}$ is Markov on $\mathbb{Z}_+$, with transition rates

$$
\begin{cases}
b^\alpha(n)p_k^\alpha(n) & \text{from } n \text{ to } n+k \text{ for all } n \geq 1 \text{ and } k \geq 1, \\
d^\alpha(n) & \text{from } n \text{ to } n-1 \text{ for all } n \geq 1, \\
0 & \text{otherwise,}
\end{cases}
$$

where $b^\alpha(n) := b_n(\alpha_n)$ and $p_k^\alpha(n) := p_{n,k}(\alpha_n)$. In other words, its infinitesimal generator is given by

$$
\mathcal{L}^\alpha u_n := b^\alpha(n) \sum_{k=1}^\infty (u_{n+k} - u_n)\, p_k^\alpha(n) + d^\alpha(n)\, (u_{n-1} - u_n), \quad (6)
$$

for all bounded real-valued sequence $(u_n)_{n\in\mathbb{Z}_+}$.

We give now a new set of assumptions that, together with Hypothesis 1, imply the existence and uniqueness of a QSD for $X^{x,\alpha}$ for any Markov control $\alpha \in \mathcal{A}_M$.

Hypothesis 2. *Assume that:*
(i) there exists $\epsilon > 0$ and $\underline{d} > 0$ such that

$$
d_n(a) \geq \underline{d} n^{1+\epsilon}, \quad \forall n \in \mathbb{Z}_+, \, a \in A;
$$

(ii) *for all $y, z \in \mathbb{N}$,*

$$\inf_{\alpha \in \mathcal{A}_M} \mathbb{P}_y \left(X_1^\alpha = z \right) > 0.$$

Point (i) implies that the absorption state 0 is accessible from any other state and any choice of the control. Further, together with the point (ii) of Hypothesis 1, it ensures that the population goes extinct almost surely. Point (ii) is an irreducibility property of the branching process, away from the absorbing point, uniformly w.r.t. the control. In particular, if we assume that for all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$$\inf_{a \in A} p_{n,k}(a) b_n(a) > 0,$$

this condition is satisfied.

Theorem 2. *Under Hypotheses 1 and 2, for all $\alpha \in \mathcal{A}_M$, there exists a unique QSD π^α on $\mathbb{N}$ for the process X^α, and there exist constants $C, \gamma > 0$ such that, for all $\alpha \in \mathcal{A}_M$ and all $x \in \mathbb{N}$,*

$$\|\mathbb{P}(X^{x,\alpha} \in \cdot \mid t < \tau^{x,\alpha}) - \pi^\alpha\|_{TV} \leq C e^{-\gamma t}, \quad \forall t \geq 0,$$

where $\|\cdot\|_{TV}$ is the total variation norm and $\tau^{x,\alpha}$ is the extinction time of $X^{x,\alpha}$, that is,

$$\tau^{x,\alpha} := \inf \left\{ s \geq 0 : \ X_s^{x,\alpha} = 0 \right\}.$$

This theorem will be proved with other properties related to QSD, gathered in Proposition 3 below, in Section 5. For all $\alpha \in \mathcal{A}_M$, we denote by $\lambda^\alpha > 0$ the absorption rate of π^α.

Proposition 3. *Under Hypotheses 1 and 2, for all $\alpha \in \mathcal{A}_M$, there exists a map η^α from $\mathbb{N}$ to $(0, +\infty)$ such that, for all $x \in \mathbb{N}$,*

$$\sup_{x \in \mathbb{N}} \left| \mathbb{P}(t < \tau^{x,\alpha}) e^{\lambda^\alpha t} - \eta^\alpha(x) \right| \leq C' e^{-\gamma t}, \tag{7}$$

where the constant C' does not depend on $\alpha \in \mathcal{A}_M$. In addition, $\pi^\alpha(\eta^\alpha) = 1$ and $\mathcal{L}^\alpha \eta^\alpha = -\lambda^\alpha \eta^\alpha$, where $\mathcal{L}^\alpha$ is the generator of X^α.

3 Main results

3.1 Optimality criterion

Let $\beta \in \mathbb{R}$ be the discount factor and $f : \mathbb{Z}_+ \times A \to \mathbb{R}$ be the running cost.

Hypothesis 3. *Assume that:*
(i) *f is measurable and non-negative;*
(ii) *f is bounded and satisfies*

$$f(0, a) = 0, \quad \forall a \in A.$$

Consider the β-discounted infinite horizon criterion given by the cost/reward function, defined for all $\alpha \in \mathcal{A}$ and $x \in \mathbb{Z}_+$, by

$$J_\beta(x,\alpha) := \mathbb{E}\left[\int_0^{+\infty} e^{\beta s} f\left(X_s^{x,\alpha}, \alpha_s\right) ds\right] = \mathbb{E}_x\left[\int_0^{+\infty} e^{\beta s} f\left(X_s^\alpha, \alpha_s\right) ds\right],$$

where we denote by $\mathbb{P}_x$ the law of $X^{x,\alpha}$ and $\mathbb{E}_x$ the corresponding expectation, and use the generic notation X^α for the process $X^{x,\alpha}$ under $\mathbb{P}_x$. Notice that the cost/reward function can also be written as

$$J_\beta(x,\alpha) = \mathbb{E}\left[\int_0^{\tau^{x,\alpha}} e^{\beta t} f\left(X_s^{x,\alpha}, \alpha_s\right) ds\right] = \mathbb{E}_x\left[\int_0^{\tau^\alpha} e^{\beta t} f\left(X_s^\alpha, \alpha_s\right) ds\right].$$

Under Hypothesis 3, the cost/reward function is well-defined and takes values in $[0, +\infty]$. The next proposition provides estimates for this function.

Proposition 4. *Under Hypotheses 1, 2 and 3, there exists a constant $C > 0$ such that, for all $\alpha \in \mathcal{A}_M$ and $\beta < \lambda^\alpha$,*

$$J_\beta(x,\alpha) \leq C\, \frac{\|f\|_\infty}{\lambda^\alpha - \beta}, \quad \forall x \in \mathbb{N}. \tag{8}$$

Proof. The Fubini-Tonelli theorem implies that

$$J_\beta(x,\alpha) = \int_0^{+\infty} e^{\beta t} \mathbb{E}_x\left[f^\alpha\left(X_t^\alpha\right)\right] dt \leq \|f\|_\infty \int_0^{+\infty} e^{\beta t} \mathbb{P}_x\left(t < \tau^\alpha\right) dt$$

$$\leq (C' + \eta^\alpha(x)) \frac{\|f\|_\infty}{\lambda^\alpha - \beta},$$

where the last inequality follows from Proposition 3. Since this proposition also implies that $\eta^\alpha(x) \leq C' + 1$ for all $x \in \mathbb{N}$ and $\alpha \in \mathcal{A}_M$, the inequality (8) is proved with $C := 2C' + 1$. $\qquad\square$

3.2 Extinction-oriented problems

In this section, we study stochastic control problems that favor extinction of the population. Since $f(0,\cdot) \equiv 0$, we aim to minimize the optimality criterion, also called cost function, given in Section 3.1.

Let us define the infinite horizon value function by

$$v_\beta(x) := \inf_{\alpha \in \mathcal{A}} J_\beta(x,\alpha).$$

Before giving the results of this section, we need to make some assumptions.

Hypothesis 4. *Assume that:*
(i) A is a compact metric space;
(ii) b_n, d_n, $p_{n,k}$ and $f(n,\cdot)$ are continuous on A for all $n \in \mathbb{N}$.

Hypothesis 5. *Assume that there exists $x \in \mathbb{N}$ such that $f(x,a) > 0$ for all $a \in A$.*

Hypothesis 4 is a classical assumption in the theory of stochastic control that ensures in particular the existence of optimal Markov controls. Note that, we could have allowed the control space $A(n)$ to depend on the position n of the process, by setting $f(n, a) = +\infty$ if $a \in A \setminus A(n)$. In this setting, our main results would hold true if Hypothesis 4 (i) is replaced by the hypothesis that $A(n)$ is a compact metric space for all $n \in \mathbb{Z}_+$. However, we restrict here to the case of a control space independent of the position of the process to keep the presentation and notations simple. Hypothesis 5 is not really restrictive. Indeed, if it is not satisfied, then the value function is identically zero since one can easily construct a Markov control such that the corresponding cost function vanishes.

The next result gives the behaviour of the value function as $\beta \uparrow \lambda^*$ where

$$\lambda^* := \sup_{\alpha \in \mathcal{A}_M} \lambda^\alpha.$$

Theorem 5. *Under Hypotheses 1, 2, 3, 4 and 5, for all $\beta < \lambda^*$, there exists an optimal Markov control for the infinite horizon problem, that is, there exists $\alpha_\beta \in \mathcal{A}_M$ such that $v_\beta = J_\beta(\cdot, \alpha_\beta)$. In addition, it holds*

$$\lim_{\substack{\beta \to \lambda^* \\ \beta < \lambda^*}} (\lambda^* - \beta) \, v_\beta(x) = \inf_{\substack{\alpha \in \mathcal{A}_M \\ \lambda^\alpha = \lambda^*}} \{\pi^\alpha(f^\alpha)\eta^\alpha(x)\},$$

where for all $\alpha \in \mathcal{A}_M$, the function f^α is defined by $f^\alpha(n) = f(n, \alpha_n)$ for all $n \in \mathbb{N}$. Further, there exists $\alpha^* \in \mathcal{A}_M$ such that $\lambda^{\alpha^*} = \lambda^*$ and

$$\inf_{\substack{\alpha \in \mathcal{A}_M \\ \lambda^\alpha = \lambda^*}} \{\pi^\alpha(f^\alpha)\eta^\alpha(x)\} = \pi^{\alpha^*}(f^{\alpha^*})\eta^{\alpha^*}(x), \quad \forall x \in \mathbb{N}.$$

The theorem above ensures that the β-discounted infinite horizon problem turns into an optimization problem on the QSD of the controlled Markov processes as $\beta \uparrow \lambda^*$. This new problem consists in minimizing the integral of the running cost w.r.t. the QSD over all the processes with the highest extinction rate. From that perspective, it is analoguous to the classical result (2). However, the optimisation problem on the QSD is not homogeneous as it depends also on the starting point through the eigenvector η^α. Yet the corresponding optimal control α^* does not depend on the starting point.

As a by-product, Theorem 5 shows that the extinction rate cannot be increased above λ^* by using non-Markov controls.

Corollary 6. *Under Hypotheses 1, 2 and 4, the population process cannot extinguish faster using a non-Markovian control than using a Markov control, in the sense that for all $\alpha \in \mathcal{A}$ and $x \in \mathbb{N}$,*

$$\int_0^{+\infty} e^{\lambda^* s} \mathbb{P}_x \left(s < \tau^\alpha\right) ds = +\infty.$$

3.3 Survival-oriented problems

In this section, we provide results similar to Section 3.2 for a control problem that favors the survival of the population. Here, we aim to maximize the optimality criterion of Section 3.1. This result can also be seen as a version of Theorem 5 for non-positive running costs (by a simple change of sign).

Let us define the infinite horizon value function by

$$w_\beta(x) := \sup_{\alpha \in \mathcal{A}} J_\beta(x, \alpha).$$

The next result is anologuous to Theorem 5 as it gives the behaviour of the value function as $\beta \uparrow \lambda_*$ where

$$\lambda_* := \inf_{\alpha \in \mathcal{A}_M} \lambda^\alpha.$$

Theorem 7. *Under Hypotheses 1, 2, 3 and 4, for all $\beta < \lambda_*$, there exist an optimal Markov control for the infinite horizon problem, and for all $x \in \mathbb{N}$,*

$$\lim_{\substack{\beta \to \lambda_* \\ \beta < \lambda_*}} (\lambda_* - \beta) w_\beta(x) = \sup_{\substack{\alpha \in \mathcal{A}_M \\ \lambda^\alpha = \lambda_*}} \{\pi^\alpha(f^\alpha)\eta^\alpha(x)\}.$$

Further, there exists $\alpha_* \in \mathcal{A}_M$ such that $\lambda^{\alpha_*} = \lambda_*$ and

$$\sup_{\substack{\alpha \in \mathcal{A}_M \\ \lambda^\alpha = \lambda_*}} \{\pi^\alpha(f^\alpha)\eta^\alpha(x)\} = \pi^{\alpha_*}(f^{\alpha_*})\eta^{\alpha_*}(x), \quad \forall x \in \mathbb{N}.$$

The last result also shows that the extinction rate cannot be reduced below λ_* by using non-Markov controls.

Corollary 8. *Under Hypotheses 1, 2 and 4, the population process cannot survive longer using a non-Markovian control, in the sense that for all $\beta < \lambda_*$ and $x \in \mathbb{N}$,*

$$\sup_{\alpha \in \mathcal{A}} \left\{ \int_0^{+\infty} e^{\beta s} \mathbb{P}_x(s < \tau^\alpha) \, ds \right\} < +\infty.$$

4 Optimal control problems

The aim of this section is to derive the HJB equations corresponding to the infinite horizon problems and to prove that, under suitable conditions, there exists optimal Markov controls. A key step in our analysis is to establish the corresponding dynamic programming principle for which we provide a new proof based on a pseudo-Markov property (see Lemma 10 below). In addition, compared to the classical results available in the literature, we also deal with the case $\beta \geq 0$.

4.1 Preliminaries

Let M be the space of integer-valued Borel measures on $\mathbb{R}_+ \times \mathbb{R}_+$ that are locally finite, *i.e.*, take finite values on bounded Borel sets. Equipped with the vague topology, it becomes a Polish space (see, *e.g.*, [Kal86, App.15.7.7]). Denote by $(\mathcal{M}_s)_{s \geq 0}$ its canonical filtration, *i.e.*, $\mathcal{M}_s$ is the smallest σ–algebra such that, for all $B \in \mathcal{B}([0, s]) \otimes \mathcal{B}(\mathbb{R}_+)$ bounded, $\mu \in M \mapsto \mu(B) \in \mathbb{Z}_+$ is measurable, or equivalently, the σ–algebra generated by $\mu \in M \mapsto \mu_{|[0,s]} \in M$.

Let us define the canonical space as follows:

$$\Omega^\circ = M \times M, \quad \mathcal{F}_s^\circ = \mathcal{M}_s \otimes \mathcal{M}_s, \quad \mathbb{P}^\circ = \mathbb{Q} \otimes \mathbb{Q},$$

where $\mathbb{Q}$ is the distribution on M of a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with Lebesgue intensity measure. We also define for all $(\mu_1, \mu_2) \in \Omega^\circ$ and $B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(R_+)$,

$$Q_1^\circ (\mu_1, \mu_2, B) = \mu_1(B), \quad Q_2^\circ (\mu_1, \mu_2, B) = \mu_2(B).$$

Our first goal in this section is to prove that the value function is invariant under change of probability space or Poisson random measures. In particular, we can assume in the sequel that we work in the canonical space, which has the desirable property of being Polish. As a first step, the next proposition provides an essential representation of admissible controls.

Proposition 9. *A process α is an admissible control if and only if there exists a process $\alpha^\circ : \mathbb{R}_+ \times \Omega^\circ \to A$ predictable w.r.t. $(\mathcal{F}_s^\circ)_{s\geq 0}$ such that for all $s \geq 0$ and $\omega \in \Omega$,*

$$\alpha_s(\omega) = \alpha_s^\circ (Q_1(\omega), Q_2(\omega)) = \alpha_s^\circ \left(Q_{1|[0,s)}(\omega), Q_{2|[0,s)}(\omega) \right). \tag{9}$$

Proof. It is clear that the process $(s, \omega) \mapsto (Q_{1|[0,s)}(\omega), Q_{2|[0,s)}(\omega))$ is predictable w.r.t. $(\mathcal{F}_s)_{s\geq 0}$. Thus, if α satisfies (9), it is an admissible control. For the converse implication, we can suppose that $A = \mathbb{R}$ since any Polish space is Borel isomorphic to a Borel subset of $\mathbb{R}$ (see, *e.g.*, [IW89]). We begin with a process α of the form $\alpha = \mathbb{1}_{(t_1,t_2] \times F}$ with $F \in \mathcal{F}_{t_1}$ and $0 < t_1 < t_2$. Since $\mathcal{F}_{t_1}$ coincides with the σ–algebra generated by $(Q_{1|[0,t_1]}, Q_{2|[0,t_1]})$, there exists $F^\circ \in \mathcal{F}^\circ$ such that

$$\alpha_s(\omega) = \mathbb{1}_{(t_1,t_2] \times F^\circ} \left(s, Q_{1|[0,t_1]}(\omega), Q_{2|[0,t_1]}(\omega) \right), \quad \forall s \geq 0, \, \omega \in \Omega.$$

Hence, the relation (9) is satisfied with

$$\alpha_s^\circ (\mu_1, \mu_2) = \mathbb{1}_{(t_1,t_2] \times F^\circ} \left(s, \mu_{1|[0,t_1]}, \mu_{2|[0,t_1]} \right), \quad \forall s \geq 0, \, \omega \in \Omega.$$

Further, α° is clearly predictable w.r.t. $(\mathcal{F}_s^\circ)_{s\geq 0}$ as a càg adapted process. The corresponding result for $\alpha = \mathbb{1}_{\{0\} \times F}$ with $F \in \mathcal{F}_0 = \{\emptyset, \Omega\}$ is obvious. Recall now that the events of the form $(t_1, t_2] \times F$ and $\{0\} \times F$ as above generate the predictable σ-algebra (see, *e.g.*, [DM75, Thm.IV.64]). Hence the conclusion follows from a monotone class argument. $\square$

Let $x \in \mathbb{Z}_+$, $\alpha \in \mathcal{A}$ and the corresponding α° given by Proposition 9 be fixed. In view of Proposition 1, there exists a unique (up to indistinguishability) adapted and càdlàg process X^{x,α° in $(\Omega^\circ, (\mathcal{F}_s^\circ)_{s\geq 0}, \mathbb{P}^\circ)$ satisfying

$$X_s^{x,\alpha^\circ} = x + \int_{(0,s] \times \mathbb{R}_+} \left(\sum_{k\geq 1} k \mathbb{1}_{I_k \left(X_{\theta-}^{x,\alpha^\circ}, \alpha_\theta^\circ \right)}(z) \right) Q_1^\circ(d\theta, dz)$$

$$- \int_{(0,s] \times \mathbb{R}_+} \mathbb{1}_{\left[0, d\left(X_{\theta-}^{x,\alpha^\circ}, \alpha_\theta^\circ \right)\right)}(z) \, Q_2^\circ(d\theta, dz), \quad \forall s \geq 0, \, \mathbb{P}^\circ - \text{a.s.}$$

Then it is clear that $X^{x,\alpha^\circ}(Q_1, Q_2)$ satisfies the relation (3) in $(\Omega, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P})$. It results from Proposition 1 that $X^{x,\alpha^\circ}(Q_1, Q_2)$ is $\mathbb{P}$–indistinguishable from $X^{x,\alpha}$. In particular, the distribution of $X^{x,\alpha}$ under $\mathbb{P}$ coincides with the distribution of

X^{x,α° under $\mathbb{P}^\circ$. As a consequence, the value function is invariant under change of probability space or Poisson random measures. In the rest of the paper, we assume that we work in the canonical space and we omit the index $\circ$ in the notations.

To conclude this section, we give a technical lemma which plays a crucial role in the proof of the dynamic programming principle (DPP for short) below: the so-called pseudo-Markov property for controlled branching processes, as it reduces to the (strong) Markov property in the uncontrolled setting. Let us start with a definition. For $\alpha \in \mathcal{A}$, we define, for all $t \geq 0$ and $\bar{\omega} \in \Omega$, the shifted control $\alpha^{t,\bar{\omega}}$ as follows:

$$\alpha_s^{t,\bar{\omega}}(\omega) := \alpha_{t+s}\left(Q_1(\bar{\omega}) \otimes_t Q_1^t(\omega), Q_2(\bar{\omega}) \otimes_t Q_2^t(\omega)\right), \quad \forall s \geq 0, \omega \in \Omega,$$

where, for $(\mu_1, \mu_2) \in \mathrm{M}^2$, $\mu_1 \otimes_t \mu_2$ denotes the element of M given by

$$\mu_1 \otimes_t \mu_2 = \mu_{1\,|[0,t]} + \mu_{2\,|(t,+\infty)},$$

and, for $\mu \in \mathrm{M}$, μ^t denotes the image measure of μ by the map $(s,z) \in \mathbb{R}_+ \times \mathbb{R}_+ \mapsto (t+s,z) \in [t,+\infty) \times \mathbb{R}_+$. In particular, $\bar{\omega}$ being fixed, it is clear that $\alpha^{t,\bar{\omega}}$ is an admissible control.

Lemma 10. *Under Hypothesis 1, for all $x \in \mathbb{Z}_+$, $\alpha \in \mathcal{A}$, $s \in \mathbb{R}_+$, τ stopping time valued in $[0,s]$ and $\varphi \in (\mathbb{R}_+)^{\mathbb{Z}_+}$, it holds*

$$\mathbb{E}\left[\varphi\left(X_s^{x,\alpha}\right)\middle|\mathcal{F}_\tau\right](\bar{\omega}) = \mathbb{E}\left[\varphi\left(X_{s-\tau(\bar{\omega})}^{X_\tau^{x,\alpha}(\bar{\omega}),\alpha^{\tau(\bar{\omega}),\bar{\omega}}}\right)\right], \quad \mathbb{P}(d\bar{\omega}) - \text{a.s.}$$

Proof. By definition of $X^{x,\alpha}$, it is clear that

$$X_s^{x,\alpha} = X_{\tau \wedge s}^{x,\alpha} + \int_{(\tau,\tau \vee s] \times \mathbb{R}_+} \left(\sum_{k \geq 1} k \mathbb{1}_{I_k\left(X_{\theta-}^{x,\alpha},\alpha_\theta\right)}(z)\right) Q_1(d\theta, dz)$$

$$- \int_{(\tau,\tau \vee s] \times \mathbb{R}_+} \mathbb{1}_{\left[0,d\left(X_{\theta-}^{x,\alpha},\alpha_\theta\right)\right)}(z)\, Q_2(d\theta, dz), \quad \forall s \geq 0, \ \mathbb{P}\text{–a.s.}$$

Since $\mathbb{P}$ is a probability measure on the Borel σ-algebra of a Polish space, there exists $(\mathbb{P}_{\bar{\omega}})_{\bar{\omega} \in \Omega}$ a family of regular conditional probabilities of $\mathbb{P}$ given $\mathcal{F}_\tau$ (see, e.g., Karatzas and Shreve [KS91, Sec.5.3.C]). It follows that, for $\mathbb{P}$–a.a. $\bar{\omega} \in \Omega$,

$$\mathbb{E}^{\mathbb{P}}\left[\varphi\left(X_s^{x,\alpha}\right)\middle|\mathcal{F}_\tau\right](\bar{\omega}) = \mathbb{E}^{\mathbb{P}_{\bar{\omega}}}\left[\varphi\left(X_s^{x,\alpha}\right)\right], \tag{10}$$

and

$$\mathbb{P}_{\bar{\omega}}\left(\tau = \tau(\bar{\omega}), X_{\tau \wedge \cdot}^{x,\alpha} = X_{\tau \wedge \cdot}^{x,\alpha}(\bar{\omega}), \alpha = \bar{\alpha}^{\tau(\bar{\omega}),\bar{\omega}}\right) = 1,$$

where

$$\bar{\alpha}_s^{\tau(\bar{\omega}),\bar{\omega}}(\omega) := \alpha_s\left(Q_1(\bar{\omega}) \otimes_{\tau(\bar{\omega})} Q_1(\omega), Q_2(\bar{\omega}) \otimes_{\tau(\bar{\omega})} Q_2(\omega)\right), \quad \forall s \geq 0, \omega \in \Omega.$$

In particular, notice that $\bar{\alpha}_{\tau(\bar{\omega})+\cdot}^{\tau(\bar{\omega}),\bar{\omega}}(Q_1^{\tau(\bar{\omega})}, Q_2^{\tau(\bar{\omega})})$ coincides with $\alpha^{\tau(\bar{\omega}),\bar{\omega}}$ by definition. We deduce that, for $\mathbb{P}$–a.a. $\bar{\omega} \in \Omega$,

$$X_s^{x,\alpha} = X_\tau^{x,\alpha}(\bar{\omega}) + \int_{(\tau(\bar{\omega}),s] \times \mathbb{R}_+} \left(\sum_{k \geq 1} k \mathbb{1}_{I_k\left(X_{\theta-}^{x,\alpha},\bar{\alpha}_\theta^{\tau(\bar{\omega}),\bar{\omega}}\right)}(z)\right) Q_1(d\theta, dz)$$

$$- \int_{(\tau(\bar{\omega}),s] \times \mathbb{R}_+} \mathbb{1}_{\left[0,d\left(X_{\theta-}^{x,\alpha},\bar{\alpha}_\theta^{\tau(\bar{\omega}),\bar{\omega}}\right)\right)}(z)\, Q_2(d\theta, dz), \quad \forall s \geq \tau(\bar{\omega}), \ \mathbb{P}_{\bar{\omega}} - \text{a.s.}$$

In other words, if we denote by μ^{-t} the image measure of $\mu_{|[t,\infty)}$ by the map $(s,z) \in [t,+\infty) \times \mathbb{R}_+ \mapsto (s-t,z) \in \mathbb{R}_+ \times \mathbb{R}_+$, it holds for $\mathbb{P}$–a.a. $\bar{\omega} \in \Omega$,

$$X^{x,\alpha}_{\tau(\bar{\omega})+s} = X^{x,\alpha}_\tau(\bar{\omega}) + \int_{(0,s]\times\mathbb{R}_+} \left(\sum_{k\geq 1} k \mathbb{1}_{I_k\left(X^{x,\alpha}_{\tau(\bar{\omega})+\theta-}, \bar{\alpha}^{\tau(\bar{\omega}),\bar{\omega}}_{\tau(\bar{\omega})+\theta}\right)}(z) \right) Q_1^{-\tau(\bar{\omega})}(d\theta, dz)$$

$$- \int_{(0,s]\times\mathbb{R}_+} \mathbb{1}_{\left[0,d\left(X^{x,\alpha}_{\tau(\bar{\omega})+\theta-}, \bar{\alpha}^{\tau(\bar{\omega}),\bar{\omega}}_{\tau(\bar{\omega})+\theta}\right)\right]}(z) Q_2^{-\tau(\bar{\omega})}(d\theta, dz), \quad \forall s \geq 0, \ \mathbb{P}_{\bar{\omega}} - \text{a.s.}$$

Hence, $X^{x,\alpha}_{\tau(\bar{\omega})+\cdot}$ is $\mathbb{P}_{\bar{\omega}}$–indistinguishable from $X^{X^{x,\alpha}_\tau(\bar{\omega}),\alpha^{\tau(\bar{\omega}),\bar{\omega}}}(Q_1^{-\tau(\bar{\omega})}, Q_2^{-\tau(\bar{\omega})})$ for $\mathbb{P}$–a.a. $\bar{\omega} \in \Omega$. In particular, the distribution of $X^{x,\alpha}_{\tau(\bar{\omega})+\cdot}$ under $\mathbb{P}_{\bar{\omega}}$ coincides with the distribution of $X^{X^{x,\alpha}_\tau(\bar{\omega}),\alpha^{\tau(\bar{\omega}),\bar{\omega}}}$ under $\mathbb{P}$ for $\mathbb{P}$–a.a. $\bar{\omega} \in \Omega$. We then conclude the proof by using (10). $\qquad \square$

4.2 Minimization problem

In this section, we derive the HJB equation corresponding to the minization problem and we prove the existence of an optimal Markov control. All these results are stated under the assumption that the value function v_β is bounded, which is implied in our case by Proposition 4 if $\beta < \lambda^*$.

Hypothesis 6. *The value function v_β is bounded on $\mathbb{Z}_+$.*

We begin with a preliminary result, which plays a crucial role in the proof of the DPP. It is a straightforward application of the pseudo-Markov property of Lemma 10.

Proposition 11. *Under Hypotheses 1, 3 (i) and 6, for all $\alpha \in \mathcal{A}$, $x \in \mathbb{Z}_+$ and τ stopping time, it holds*

$$J_\beta(x,\alpha) = \mathbb{E}\left[\int_0^\tau e^{\beta s} f\left(X^{x,\alpha}_s, \alpha_s\right) ds + e^{\beta\tau} J_\beta\left(X^{x,\alpha}_\tau, \alpha^\tau\right) \right],$$

where we mean by the notation $\mathbb{E}\left[e^{\beta\tau} J_\beta\left(X^{x,\alpha}_\tau, \alpha^\tau\right)\right]$ the quantity

$$\int_\Omega e^{\beta\tau(\omega)} J_\beta\left(X^{x,\alpha}_\tau(\omega), \alpha^{\tau(\omega),\omega}\right) \mathbb{P}(d\omega).$$

Proof. Lemma 10 yields

$$\mathbb{E}\left[\int_\tau^{+\infty} e^{\beta s} f\left(X^{x,\alpha}_s, \alpha_s\right) ds \, \Big| \, \mathcal{F}_\tau \right](\omega) = e^{\beta\tau(\omega)} J_\beta\left(X^{x,\alpha}_\tau(\omega), \alpha^{\tau(\omega),\omega}\right), \quad \mathbb{P}(d\omega) - \text{a.s.}$$

The conclusion follows from conditioning by $\mathcal{F}_\tau$ in the cost function. $\qquad \square$

We are now in a position to give the DPP, which is the keytone of our analysis of this optimal stochastic control problem.

Theorem 12. *Under Hypotheses 1, 3 (i) and 6, it holds for all $x \in \mathbb{Z}_+$ and τ bounded stopping time,*

$$v_\beta(x) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^\tau e^{\beta s} f\left(X^{x,\alpha}_s, \alpha_s\right) ds + e^{\beta\tau} v_\beta\left(X^{x,\alpha}_\tau\right) \right].$$

Proof. In view of Proposition 11, it is clear that

$$J_\beta(x, \alpha) \geq \mathbb{E}\left[\int_0^\tau e^{\beta s} f\left(X_s^{x,\alpha}, \alpha_s\right) ds + e^{\beta\tau} v_\beta\left(X_\tau^{x,\alpha}\right)\right].$$

Hence, we conclude that

$$v_\beta(x) \geq \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^\tau e^{\beta s} f\left(X_s^{x,\alpha}, \alpha_s\right) ds + e^{\beta\tau} v_\beta\left(X_\tau^{x,\alpha}\right)\right].$$

The reverse inequality is more difficult to derive. The idea is to construct a convenient control by concatenation of an arbitrary control on $[0, \tau]$ and a well-chosen control on $[\tau, \infty)$. We start by working with a simple stopping time, *i.e.*, taking values in a countable subset on $\mathbb{R}_+$. Given $\varepsilon > 0$, let α^y be an ε–optimal control for the discount factor β and the initial state $y \in \mathbb{Z}_+$, *i.e.*,

$$v_\beta(y) \geq J_\beta(y, \alpha^y) - \varepsilon.$$

Let also α be an arbitrary admissible control. Define the control $\tilde{\alpha}$ by

$$\tilde{\alpha}_s(\omega) = \begin{cases} \alpha_s(\omega), & \text{if } 0 \leq s \leq \tau(\omega), \\ \alpha_{s-t}^{X_\tau^{x,\alpha}(\omega)}(Q_1^{-\tau(\omega)}(\omega), Q_2^{-\tau(\omega)}(\omega)), & \text{if } s > \tau(\omega), \end{cases}$$

where, for $\mu \in \mathrm{M}$, μ^{-t} denotes the image measure of $\mu_{|[t,\infty)}$ by the map $(s, z) \in [t, +\infty) \times \mathbb{R}_+ \mapsto (s - t, z) \in \mathbb{R}_+ \times \mathbb{R}_+$. Since τ is a simple stopping time, it is clear that the process $\tilde{\alpha}$ is predictable and thus an admissible control. By definition, $\tilde{\alpha}^{\tau(\omega),\omega}$ coincides with $\alpha^{X_\tau^{x,\alpha}(\omega)}$ and we have

$$\mathbb{E}\left[e^{\beta\tau} v_\beta\left(X_\tau^{x,\alpha}\right)\right] \geq \mathbb{E}\left[e^{\beta\tau} J_\beta\left(X_\tau^{x,\alpha}, \alpha^{X_\tau^{x,\alpha}}\right)\right] - \varepsilon\mathbb{E}\left[e^{\beta\tau}\right]$$
$$\geq \mathbb{E}\left[e^{\beta\tau} J_\beta\left(X_\tau^{x,\alpha}, \tilde{\alpha}^\tau\right)\right] - \varepsilon e^{\beta\|\tau\|_\infty}.$$

Then it follows from Proposition 11 that

$$\mathbb{E}\left[\int_0^\tau e^{\beta s} f\left(X_s^{x,\alpha}, \alpha_s\right) ds + e^{\beta\tau} v_\beta\left(X_\tau^{x,\alpha}\right)\right] \geq J_\beta(x, \tilde{\alpha}) - \varepsilon e^{\beta\|\tau\|_\infty}$$
$$\geq v_\beta(x) - \varepsilon e^{\beta\|\tau\|_\infty}.$$

Sending ε to zero and taking the infimum over $\alpha \in \mathcal{A}$, we conclude that

$$v_\beta(x) \leq \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^\tau e^{\beta s} f\left(X_s^{x,\alpha}, \alpha_s\right) ds + e^{\beta\tau} v_\beta\left(X_\tau^{x,\alpha}\right)\right].$$

To conclude, it remains to derive the same result for any bounded stopping time. To this end, we consider $(\tau_n)_{n \in \mathbb{N}}$ a decreasing sequence of simple stopping times converging to τ, *e.g.*,

$$\tau_n = \sum_{i \geq 0} \frac{i+1}{2^n} \mathbb{1}_{\frac{i}{2^n} < \tau \leq \frac{i+1}{2^n}}.$$

In view of the above, we have for all $n \in \mathbb{N}$,

$$\mathbb{E}\left[\int_0^{\tau_n} e^{\beta s} f\left(X_s^{x,\alpha}, \alpha_s\right) ds + e^{\beta\tau_n} v_\beta\left(X_{\tau_n}^{x,\alpha}\right)\right] \geq v_\beta(x) - \varepsilon e^{\beta\|\tau_n\|_\infty}.$$

Since v_β is bounded by assumption, the conclusion follows by applying the dominated convergence theorem as n tends to infinity. $\qquad\square$

From the DPP, we derive in the next theorem the HJB equation corresponding to the minimization problem.

Theorem 13. *Under Hypotheses 1, 3 (i) and 6, it holds*

$$\beta v_\beta(x) + \inf_{a \in A} \{\mathcal{L}^a v_\beta(x) + f(x,a)\} = 0, \quad \forall\, x \in \mathbb{N}. \tag{11}$$

This result can be deduced from the DPP by classical arguments. We give its proof for the sake of completeness.

Proof. Let $x \in \mathbb{N}$ be fixed. Denote by τ_h the stopping time given by

$$\tau_h := \inf\left\{s \geq 0;\ Q_1([0,s] \times [0,\bar{b}x]) + Q_2([0,s] \times [0,\bar{d}_x]) \neq 0\right\} \wedge h,$$

for some $h > 0$. First, we notice that

$$
\begin{aligned}
e^{\beta \tau_h} v_\beta\left(X_{\tau_h}^{x,\alpha}\right) = {}& v_\beta(x) + \int_0^{\tau_h} \beta e^{\beta\theta} v_\beta(x)\, d\theta \\
& + \int_{[0,\tau_h]\times\mathbb{R}_+} \sum_{k\geq 1} \left(e^{\beta\theta} v_\beta(x+k) - e^{\beta\theta} v_\beta(x)\right) \mathbb{1}_{I_k(x,\alpha_\theta)}(z)\, Q_1(d\theta, dz) \\
& + \int_{[0,\tau_h]\times\mathbb{R}_+} \left(e^{\beta\theta} v_\beta(x-1) - e^{\beta\theta} v_\beta(x)\right) \mathbb{1}_{[0,d(x,\alpha_\theta)]}(z)\, Q_2(d\theta, dz).
\end{aligned}
$$

Taking expectation, it follows from the boundedness of v_β that

$$\mathbb{E}\left[e^{\beta\tau_h} v_\beta\left(X_{\tau_h}^{x,\alpha}\right)\right] = v_\beta(x) + \mathbb{E}\left[\int_0^{\tau_h} e^{\beta\theta}\left(\beta v_\beta(x) + \mathcal{L}^{\alpha_\theta} v_\beta(x)\right) d\theta\right].$$

Using the DPP, we deduce that

$$\inf_{\alpha \in \mathcal{A}} \left\{\mathbb{E}\left[\int_0^{\tau_h} e^{\beta\theta}\left(\beta v_\beta(x) + \mathcal{L}^{\alpha_\theta} v_\beta(x) + f(x,\alpha_\theta)\right) d\theta\right]\right\} = 0.$$

Since constant controls are admissible, the identity above is equivalent to

$$\left(\beta v_\beta(x) + \inf_{a \in A}\{\mathcal{L}^a v_\beta(x) + f(x,a)\}\right) \mathbb{E}\left[\frac{1}{h}\int_0^{\tau_h} e^{\beta\theta}\, d\theta\right] = 0.$$

Sending h to zero, we conclude by the dominated convergence theorem. □

Theorem 14. *Under Hypotheses 1, 3 (i), 4 and 6, there exists $\alpha_\beta \in \mathcal{A}_M$ such that*

$$\inf_{a \in A}\{\mathcal{L}^a v_\beta(x) + f(x,a)\} = \mathcal{L}^{\alpha_\beta(x)} v_\beta(x) + f(x, \alpha_\beta(x)).$$

In addition, the Markov control α_β is optimal for the β–discounted minization problem on infinite horizon, that is, $v_\beta = J_\beta(\cdot, \alpha_\beta)$.

Proof. Let us show first that the map $a \mapsto \mathcal{L}^a v_\beta(x) + f(x,a)$ is continuous on A for all $x \in \mathbb{Z}_+$. In view of the continuity of $\gamma(x,\cdot)$, $f(x,\cdot)$, $p_k(x,\cdot)$ and $d(x,\cdot)$, it is clearly enough to prove that

$$\lim_{K \to \infty} \sup_{a \in A} \left\{\sum_{k \geq K+1} v_\beta(x+k)\, p_k(x,a)\right\} = 0.$$

The former follows from

$$\sum_{k \geq K+1} v_\beta (x+k) p_k (x,a) \leq \|v_\beta\|_\infty \sum_{k \geq K+1} p_k (x,a) \leq \frac{M\|v_\beta\|_\infty}{K+1},$$

where the constant M comes from Hypothesis 1 (iv). Since A is compact, we deduce that there exists $\alpha_\beta \in \mathcal{A}_M$ such that

$$\inf_{a \in A} \{\mathcal{L}^a v_\beta(x) + f(x,a)\} = \mathcal{L}^{\alpha_\beta(x)} v_\beta(x) + f(x, \alpha_\beta(x)).$$

The rest of the proof relies on a classical verification theorem. Using the boundedness of v_β, we derive as in the proof of Theorem 13 that

$$\mathbb{E}\left[e^{\beta T} v_\beta \left(X_T^{x,\alpha_\beta}\right)\right] = v_\beta(x) + \mathbb{E}\left[\int_0^T e^{\beta \theta} \left(\beta v_\beta \left(X_\theta^{x,\alpha_\beta}\right) + \mathcal{L}^{\alpha_\beta} v_\beta \left(X_\theta^{x,\alpha_\beta}\right)\right) d\theta\right].$$

Since v_β is a solution of the HJB equation, we deduce that

$$\mathbb{E}\left[\int_0^T e^{\beta \theta} f^{\alpha_\beta} \left(X_\theta^{x,\alpha_\beta}\right) d\theta + e^{\beta T} v_\beta \left(X_T^{x,\alpha_\beta}\right)\right] = v_\beta(x).$$

In particular, we have

$$\mathbb{E}\left[\int_0^T e^{\beta \theta} f^{\alpha_\beta} \left(X_\theta^{x,\alpha_\beta}\right) d\theta\right] \leq v_\beta(x).$$

Sending T to infinity, it follows from the monotone convergence theorem that

$$\mathbb{E}\left[\int_0^{+\infty} e^{\beta \theta} f^{\alpha_\beta} \left(X_\theta^{x,\alpha_\beta}\right) d\theta\right] \leq v_\beta(x).$$

We conclude that α_β is an optimal control. $\qquad\square$

Remark. Our analysis can be extended to deal with the so-called continuous-time Markov decision processes (CTMDP) (see, *e.g.*, [GHL09, PZ13]). As such, this section presents a new methodology to derive the HJB equation and the existence of an optimal Markov control. Even though they can be weakened, our assumptions are rather strong compared to the literature on the CTMDP. The benefit of this approach is to unify the treatment of stochastic control problems for jump processes and diffusion processes. Indeed, by a simple analogy which consists in giving to the Brownian motion the role played by the Poisson random measures, we can follow the same arguments to derive the DPP corresponding to a stochastic control problem on diffusion processes. See [CTT16] for more details.

4.3   Maximisation problem

In this section, we derive the corresponding results for the maximization problem. They are stated under the assumption that the value function w_β is bounded, which is proved for $\beta < \lambda_*$ in Lemma 21 below. Note that this is a delicate issue in this setting.

Hypothesis 7. *The value function w_β is bounded on $\mathbb{Z}_+$.*

Theorem 15. *Under Hypotheses 1, 3 (i), 4 and 7, it holds*

$$\beta w_\beta(x) + \sup_{a \in A} \{ \mathcal{L}^a w_\beta(x) + f(x, a) \} = 0, \quad \forall x \in \mathbb{N}, \tag{12}$$

and there exists $\alpha_\beta \in \mathcal{A}_M$ such that

$$\sup_{a \in A} \{ \mathcal{L}^a w_\beta(x) + f(x, a) \} = \mathcal{L}^{\alpha_\beta(x)} w_\beta(x) + f(x, \alpha_\beta(x)).$$

If we assume further that

$$\lim_{T \to +\infty} e^{\beta T} \mathbb{E}_x \left[w_\beta \left(X_T^{\alpha_\beta} \right) \right] = 0, \quad \forall x \in \mathbb{N}, \tag{13}$$

then the Markov control α_β is optimal for the β–discounted maximization problem on infinite horizon, that is, $w_\beta = J_\beta(\cdot, \alpha_\beta)$.

Proof. The proof relies on the arguments developed in the previous section. Indeed, it follows by repeating the argument of Theorem 12 that the corresponding DPP holds, *i.e*, for any bounded stopping time τ,

$$w_\beta(x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^\tau e^{\beta s} f \left(X_s^{x,\alpha}, \alpha_s \right) ds + e^{\beta \tau} w_\beta \left(X_\tau^{x,\alpha} \right) \right].$$

Then we deduce as in Theorem 13 that the value function satisfies the HJB equation (12). To conclude, it remains to adapt the arguments of Theorem 15. First, the existence of α_β follows from the continuity of the map $a \mapsto \mathcal{L}^a w_\beta(x) + f(x, a)$, which results from the boundedness of the value function as before. Second, the proof of the verification theorem needs to be slighty modified. Since w_β is a solution of the HJB equation, we have

$$\mathbb{E} \left[\int_0^T e^{\beta \theta} f^{\alpha_\beta} \left(X_\theta^{x,\alpha_\beta} \right) d\theta + e^{\beta T} w_\beta \left(X_T^{x,\alpha_\beta} \right) \right] = w_\beta(x).$$

Sending T to infinity and using (13), it follows from the monotone convergence theorem that

$$\mathbb{E} \left[\int_0^{+\infty} e^{\beta \theta} f^{\alpha_\beta} \left(X_\theta^{x,\alpha_\beta} \right) d\theta \right] = w_\beta(x).$$

We conclude that α_β is an optimal control. $\qquad \square$

5 Quasi-stationary distributions

We fix $\alpha \in \mathcal{A}_M$ in all this section. Our goal is to study the QSD of the Markov process X^α, that is the probability measure π^α on $\mathbb{N}$ such that

$$\mathbb{P}_{\pi^\alpha}(X_t^\alpha \in \Gamma \mid t < \tau^\alpha) = \pi^\alpha(\Gamma),$$

where $\mathbb{P}_{\pi^\alpha} = \sum_{k \geq 1} \pi_k^\alpha \mathbb{P}_k$, and the associated absorption rate $\lambda^\alpha > 0$ defined by

$$\mathbb{P}_{\pi^\alpha}(t < \tau^\alpha) = e^{-\lambda^\alpha t}, \quad \forall t \geq 0. \tag{14}$$

Below, we prove Theorem 2 and auxiliary estimates needed for Proposition 3 and for the proof of our main results. In particular, throughout this section, we work under Hypotheses 1 and 2.

Since the process X^α is Markov, a.s. absorbed in finite time in 0, and satisfies

$$\mathbb{P}_x(t < \tau^\alpha) > 0, \quad \forall\, x \geq 1 \text{ and } t > 0,$$

we can apply the general criterion of [CV15a, Thm. 2.1] to prove the existence and uniqueness of a QSD π^α and the existence of constants C and $\gamma > 0$ such that, for all probability measure μ on $\mathbb{N}$,

$$\|\mathbb{P}_\mu(X^\alpha \in \cdot \mid t < \tau^\alpha) - \pi^\alpha\|_{TV} \leq Ce^{-\gamma t}, \quad \forall t \geq 0. \tag{15}$$

These three properties are equivalent to the following condition, which also implies several other properties including those of Proposition 3.

Condition (A) There exists a probability measure ν on $\mathbb{N}$ such that

(A1) there exists $t_0, c_1 > 0$ such that for all $x \in \mathbb{N}$,

$$\mathbb{P}_x(X^\alpha_{t_0} \in \cdot \mid t_0 < \tau^\alpha) \geq c_1 \nu(\cdot);$$

(A2) there exists $c_2 > 0$ such that for all $x \in \mathbb{N}$ and $t \geq 0$,

$$\mathbb{P}_\nu(t < \tau^\alpha) \geq c_2 \mathbb{P}_x(t < \tau^\alpha).$$

Proof of Theorem 2 and Proposition 3. The next lemma, proved at the end of this section, allows to check Conditions (A1) and (A2) with $\nu = \delta_1$.

Lemma 16. *For all $\alpha \in \mathcal{A}_M$, one has:*
(i) *there exists $t_0, c_1 > 0$ such that for all $x \in \mathbb{N}$,*

$$\mathbb{P}_x\left(X^\alpha_{t_0} = 1 \mid t_0 < \tau^\alpha\right) \geq c_1;$$

(ii) *there exists $c_2 > 0$ such that for all $x \in \mathbb{N}$ and $t \geq 0$,*

$$\mathbb{P}_1\left(t < \tau^\alpha\right) \geq c_2 \mathbb{P}_x\left(t < \tau^\alpha\right).$$

In addition, the constants t_0, c_1 and c_2 do not depend on $\alpha \in \mathcal{A}_M$.

By [CV15a, Thm. 2.1], these properties directly imply (15) and hence Theorem 2 with $\gamma = -\log(1 - c_1 c_2)/t_0$ and $C = 2/(1 - c_1 c_2)$. Since t_0, c_1 and c_2 do not depend on α, so do C and γ.

The existence of the function η^α of Proposition 3, limit of $\mathbb{P}_\cdot(t < \tau^\alpha)e^{\lambda^\alpha t}$ and eigenfunction of $\mathcal{L}^\alpha$, is given by [CV15a, Prop. 2.3]. In addition, the proof of Proposition 2.3 in [CV15a, Sec.5.2] implies that (7) holds with the constant $C' = C\kappa^\alpha$, where

$$\kappa^\alpha := \sup_{t \geq 0,\, \mu \in \mathcal{M}_1(\mathbb{N})} e^{\lambda^\alpha t} \mathbb{P}_\mu(t < \tau^\alpha),$$

where $\mathcal{M}_1(\mathbb{N})$ is the set of probability measures on $\mathbb{N}$. It only remains to check that κ^α is bounded as a function of $\alpha \in \mathcal{A}_M$. It follows from [CV15a, Remark 1] that

$$\kappa^\alpha \leq \inf_{s > 0} \exp\left(\lambda^\alpha s + \frac{Ce^{(\lambda^\alpha - \gamma)s}}{1 - e^{-\gamma s}}\right),$$

where C and γ are the constants of Theorem 2. So the proof will be completed if we check that λ^α is uniformly bounded w.r.t. α. This follows from the general fact that, for any Markov process in $\mathbb{Z}_+$ absorbed at 0, the extinction time is larger than an exponential random variable of parameter the supremum of the extinction rate in the population from any state in $\mathbb{N}$. In our branching process, this supremum is $d_1^\alpha \leq \bar{d}_1$. Hence, it follows from (14) that $\lambda^\alpha \leq \bar{d}_1$, which ends the proof of Proposition 3. $\qquad\square$

In order to prove Lemma 16, we need to prove that the controlled process *comes down from infinity* [vD91, CV15a]. The next lemma, based on similar arguments as in [CV15b], is a preliminary step for this.

Lemma 17. *For all $\lambda > 0$, there exists $x_\lambda \in \mathbb{N}$ such that*

$$\sup_{\substack{x \geq x_\lambda \\ \alpha \in \mathcal{A}_M}} \mathbb{E}_x \left[e^{\lambda \zeta_\lambda^\alpha} \right] < +\infty,$$

where ζ_λ^α denotes the first hitting time of $\{x_\lambda\}$ by X^α, that is,

$$\zeta_\lambda^\alpha := \inf \{ t \geq 0 : \ X_t^\alpha = x_\lambda \}.$$

Proof. Let $\psi : \mathbb{Z}_+ \to \mathbb{R}_+$ be given by

$$\psi(x) := \sum_{y=1}^{x} \frac{1}{y^{1+\frac{\epsilon}{2}}}, \quad \forall\, x \in \mathbb{Z}_+,$$

where $\epsilon > 0$ comes from Hypothesis 2 (i). We start by proving that, for all $\lambda > 0$, there exists $x_\lambda \in \mathbb{N}$ such that

$$\mathcal{L}^\alpha \psi(x) \leq -\lambda \psi(x), \quad \forall\, x \geq x_\lambda, \alpha \in \mathcal{A}_M. \tag{16}$$

It follows from Hypotheses 1 and 2 that

$$\mathcal{L}^\alpha \psi(x) \leq \underline{d} x^{1+\epsilon} \left(\psi(x-1) - \psi(x) \right) + \bar{b} x \sum_{k=1}^{+\infty} \left(\psi(x+k) - \psi(x) \right) p_k^\alpha(x)$$

$$\leq -\underline{d} x^{\frac{\epsilon}{2}} + \bar{b} x \sum_{k=1}^{+\infty} \left(\sum_{y=x+1}^{x+k} \frac{1}{y^{1+\frac{\epsilon}{2}}} \right) p_k^\alpha(x).$$

Further, one has

$$\sum_{k=1}^{+\infty} \left(\sum_{y=x+1}^{x+k} \frac{1}{y^{1+\frac{\epsilon}{2}}} \right) p_k^\alpha(x) \leq \frac{1}{x^{1+\frac{\epsilon}{2}}} \sum_{k=1}^{+\infty} k p_k^\alpha(x) \leq \frac{M}{x^{1+\frac{\epsilon}{2}}},$$

where $M > 0$ comes from Hypothesis 1 (iv). Hence, we deduce that there exists x_1 such that

$$\mathcal{L}^\alpha \psi(x) \leq -\frac{\underline{d}}{2} x^{\frac{\epsilon}{2}}, \quad \forall\, x \geq x_1.$$

Now, given $\lambda > 0$, we take x_λ as the smallest integer not less than x_1 and $\left(\frac{2\lambda \psi(\infty)}{\underline{d}} \right)^{\frac{2}{\epsilon}}$, where $\psi(\infty)$ denotes the limit of $\psi(x)$. Then one has for all $x \geq x_\lambda$,

$$\mathcal{L}^\alpha \psi(x) \leq -\frac{\underline{d}}{2} x_\lambda^{\frac{\epsilon}{2}} \leq -\frac{\underline{d} x_\lambda^{\frac{\epsilon}{2}}}{2\psi(\infty)} \psi(x) \leq -\lambda \psi(x).$$

We are now in a position to complete the proof. As already observed, we have

$$
e^{\lambda s} \psi\left(X_s^{x,\alpha}\right) = \psi(x) + \int_0^s e^{\lambda \theta} \lambda \psi\left(X_\theta^{x,\alpha}\right) d\theta
$$

$$
+ \int_{(t,s] \times \mathbb{R}_+} e^{\lambda \theta} \sum_{k=1}^{+\infty} \left(\psi\left(X_{\theta-}^{x,\alpha} + k\right) - \psi\left(X_{\theta-}^{x,\alpha}\right)\right) \mathbb{1}_{I_k^\alpha\left(X_{\theta-}^{x,\alpha}\right)}(z)\, Q_1(d\theta, dz)
$$

$$
+ \int_{(t,s] \times \mathbb{R}_+} e^{\lambda \theta} \left(\psi\left(X_{\theta-}^{x,\alpha} - 1\right) - \psi\left(X_{\theta-}^{x,\alpha}\right)\right) \mathbb{1}_{\left[0, d^\alpha\left(X_{\theta-}^{x,\alpha}\right)\right)}(z)\, Q_2(d\theta, dz).
$$

We localize the relation above by ζ_λ^α and we take the expectation. Since ψ is bounded, we obtain

$$
\mathbb{E}_x\left[e^{\lambda \zeta_\lambda^\alpha \wedge s} \psi\left(X_{\zeta_\lambda^\alpha \wedge s}^\alpha\right)\right] = \psi(x) + \mathbb{E}_x\left[\int_0^{\zeta_\lambda^\alpha \wedge s} e^{\lambda \theta}\left(\lambda \psi\left(X_\theta^\alpha\right) + \mathcal{L}^{\alpha_\theta} \psi\left(X_\theta^\alpha\right)\right) d\theta\right].
$$

It follows from the inequality (16) that

$$
\mathbb{E}_x\left[e^{\lambda \zeta_\lambda^\alpha \wedge s} \psi\left(X_{\zeta_\lambda^\alpha \wedge s}^\alpha\right)\right] \leq \psi(x), \quad \forall x \geq x_\lambda, \alpha \in \mathcal{A}_M.
$$

Hence, we deduce that

$$
\mathbb{E}_x\left[e^{\lambda \zeta_\lambda^\alpha \wedge s}\right] \leq \frac{\psi(\infty)}{\psi(x_\lambda)}, \quad \forall x \geq x_\lambda, \alpha \in \mathcal{A}_M.
$$

The conclusion follows from the monotone convergence theorem. $\qquad\square$

Proof of Lemma 16. This proof uses a similar method as in [CV15a, Thm. 4.1]. Let $\lambda = \bar{d}_1 + 1$ and $x_\lambda \in \mathbb{N}$ be given by Lemma 17 such that

$$
C_1 := \sup_{\substack{x \geq x_\lambda \\ \alpha \in \mathcal{A}_M}} \mathbb{E}_x\left[e^{\lambda \zeta_\lambda^\alpha}\right] < +\infty.
$$

We start by proving (ii). The Markov property yields, for all $x \in \mathbb{N}$,

$$
\mathbb{P}_1\left(X_1^\alpha = x\right) \mathbb{P}_x\left(t < \tau^\alpha\right) \leq \mathbb{P}_1\left(t + 1 < \tau^\alpha\right) \leq \mathbb{P}_1\left(t < \tau^\alpha\right).
$$

Hence, if we denote

$$
\frac{1}{C_2} := \inf_{\substack{\alpha \in \mathcal{A}_M \\ 1 \leq y,z \leq x_\lambda}} \mathbb{P}_y\left(X_1^\alpha = z\right),
$$

we deduce that for all $1 \leq x \leq x_\lambda$,

$$
\mathbb{P}_x\left(t < \tau^\alpha\right) \leq C_2 \mathbb{P}_1\left(t < \tau^\alpha\right). \tag{17}
$$

Note that C_2 is well-defined in view of Hypothesis 2 (ii). Further, we have for all $x \geq x_\lambda$,

$$
\mathbb{P}_x\left(t < \tau^\alpha\right) = \mathbb{P}_x\left(t < \zeta_\lambda^\alpha\right) + \mathbb{P}_x\left(\zeta_\lambda^\alpha < t < \tau^\alpha\right).
$$

By Markov's inequality, the first term on the r.h.s. above can be bounded as follows:

$$
\mathbb{P}_x\left(t < \zeta_\lambda^\alpha\right) \leq C_1 e^{-\lambda t} \leq C_1 \mathbb{P}_1\left(t < \tau^\alpha\right),
$$

where the last inequality comes from $\lambda > \bar{d}_1$. For the second term, the Markov property and the inequality (17) yield

$$\mathbb{P}_x \left(\zeta_\lambda^\alpha < t < \tau^\alpha \right) = \int_0^t \mathbb{P}_{x_\lambda} \left(t - s < \tau^\alpha \right) \mathbb{P}_x \left(\zeta_\lambda^\alpha \in ds \right)$$

$$\leq C_2 \int_0^t \mathbb{P}_1 \left(t - s < \tau^\alpha \right) \mathbb{P}_x \left(\zeta_\lambda^\alpha \in ds \right).$$

Moreover, since $\lambda > \bar{d}_1$, we have

$$\mathrm{e}^{-\lambda s} \mathbb{P}_1 \left(t - s < \tau^\alpha \right) \leq \inf_{x \geq 1} \left\{ \mathbb{P}_x \left(s < \tau^\alpha \right) \right\} \mathbb{P}_1 \left(t - s < \tau^\alpha \right) \leq \mathbb{P}_1 \left(t < \tau^\alpha \right).$$

Hence, we obtain

$$\mathbb{P}_x \left(\zeta_\lambda^\alpha < t < \tau^\alpha \right) \leq C_2 \mathbb{P}_1 \left(t < \tau^\alpha \right) \mathbb{E}_x \left[\mathrm{e}^{\lambda \zeta_\lambda^\alpha} \right] \leq C_1 C_2 \mathbb{P}_1 \left(t < \tau^\alpha \right).$$

This ends the proof of the assertion (ii).

Now we prove the assertion (i). Let λ, x_λ, C_1 and C_2 be defined as above. For all $t \geq 0$ and $\alpha \in \mathcal{A}_M$, we consider two cases: either $x \leq x_\lambda$,

$$\mathbb{P}_x \left(X_{t+1}^\alpha = 1 \right) \geq \mathbb{P}_x \left(X_1^\alpha = 1 \right) \mathbb{P}_1 \left(X^\alpha \text{ has no jump in } (0, t] \right) \geq \frac{1}{C_2} \mathrm{e}^{-(\bar{d}_1 + \bar{b})t},$$

or $x > x_\lambda$,

$$\mathbb{P}_x \left(X_{t+1}^\alpha = 1 \right) \geq \mathbb{P}_x \left(\zeta_\lambda^\alpha \leq t \right) \mathbb{P}_{x_\lambda} \left(X_1^\alpha = 1 \right) \mathbb{P}_1 \left(X^\alpha \text{ has no jump in } (0, t] \right)$$

$$\geq \left(1 - C_1 \mathrm{e}^{-\lambda t} \right)^+ \frac{1}{C_2} \mathrm{e}^{-(\bar{d}_1 + \bar{b})t}.$$

Hence, if we take $t_0 > 1$ such that $C_1 \mathrm{e}^{-\lambda(t_0 - 1)} \leq \frac{1}{2}$, we obtain

$$\mathbb{P}_x \left(X_{t_0}^\alpha = 1 \right) \geq \frac{\mathrm{e}^{-(\bar{d}_1 + \bar{b})(t_0 - 1)}}{2 C_2} > 0, \quad \forall x \in \mathbb{N}.$$

We finally notice that

$$\mathbb{P}_x \left(X_{t_0}^\alpha = 1 \mid t_0 < \tau^\alpha \right) \geq \mathbb{P}_x \left(X_{t_0}^\alpha = 1 \right), \quad \forall x \in \mathbb{N},$$

which concludes the proof of Lemma 16. $\qquad \square$

We conclude this section with a proposition, which is used in the proof of Theorems 5 and 7.

Proposition 18. *For all $x \in \mathbb{N}$, it holds*

$$\inf_{\alpha \in \mathcal{A}_M} \pi^\alpha(x) > 0, \qquad \inf_{\alpha \in \mathcal{A}_M} \eta^\alpha(x) > 0.$$

Proof. Fix $x \in \mathbb{N}$. A straightforward extension of Lemma 16 yields
(i) there exists $t_0, c_1 > 0$ such that for all $y \in \mathbb{N}$ and $\alpha \in \mathcal{A}_M$,

$$\mathbb{P}_y \left(X_{t_0}^\alpha = x \mid t_0 < \tau^\alpha \right) \geq c_1;$$

(ii) there exists $c_2 > 0$ such that for all $y \in \mathbb{N}$, $t \geq 0$ and $\alpha \in \mathcal{A}_M$,

$$\mathbb{P}_x \left(t < \tau^\alpha \right) \geq c_2 \mathbb{P}_y \left(t < \tau^\alpha \right).$$

Note that the constants t_0, c_1 and c_2 above depend on x. Then, we integrate the first inequality w.r.t. π^α. It yields

$$\pi^\alpha(x) \geq c_1, \quad \forall \alpha \in \mathcal{A}_M.$$

Finally, we integrate the second inequality w.r.t. π^α, multiply it by $e^{\lambda^\alpha t}$ and send t to infinity. We obtain

$$\eta^\alpha(x) \geq c_2, \quad \forall \alpha \in \mathcal{A}_M. \qquad \square$$

6 Proof of the main results

6.1 Proof of Theorem 5

We begin with a premilinary result, which is a refinement of Proposition 4.

Proposition 19. *Under Hypotheses 1, 2 and 3, there exists $C > 0$ such that, for all $\alpha \in \mathcal{A}_M$ satisfying $\lambda^\alpha > \beta$, for all $x \in \mathbb{N}$,*

$$\left| J_\beta(x, \alpha) - \frac{\pi^\alpha(f^\alpha)}{\lambda^\alpha - \beta} \eta^\alpha(x) \right| \leq C.$$

In addition, under Hypotheses 4 and 5, for all $\alpha \in \mathcal{A}_M$ such that $\lambda^\alpha \leq \beta$, it holds $J_\beta(x, \alpha) = +\infty$ for all $x \in \mathbb{N}$.

Proof. First we observe that

$$\left| \mathbb{E}_x \left[f^\alpha \left(X_t^\alpha \right) \right] - \pi^\alpha(f^\alpha) \eta^\alpha(x) e^{-\lambda^\alpha t} \right|$$
$$\leq \eta^\alpha(x) e^{-\lambda^\alpha t} \left| \mathbb{E}_x \left[f^\alpha \left(X_t^\alpha \right) \mid t < \tau^\alpha \right] - \pi^\alpha(f^\alpha) \right|$$
$$+ \|f\|_\infty \left| \mathbb{P}_x \left(t < \tau^\alpha \right) - \eta^\alpha(x) e^{-\lambda^\alpha t} \right|.$$

On the one hand, we deduce from Theorem 2 that

$$\left| \mathbb{E}_x \left[f^\alpha \left(X_t^\alpha \right) \mid t < \tau^\alpha \right] - \pi^\alpha(f^\alpha) \right| \leq \|f\|_\infty \| \mathbb{P}_x \left(X_t^\alpha \in \cdot \mid t < \tau^\alpha \right) - \pi^\alpha \|_{TV}$$
$$\leq C \|f\|_\infty e^{-\gamma t}.$$

On the other hand, it follows from Proposition 3 that for all $x \in \mathbb{N}$ and $\alpha \in \mathcal{A}_M$,

$$\eta^\alpha(x) \leq C' + 1 \quad \text{and} \quad \left| \mathbb{P}_x \left(t < \tau^\alpha \right) - \eta^\alpha(x) e^{-\lambda^\alpha t} \right| \leq C' e^{-(\lambda^\alpha + \gamma)t}.$$

Hence, we deduce that

$$\left| \mathbb{E}_x \left[f^\alpha \left(X_t^\alpha \right) \right] - \pi^\alpha(f^\alpha) \eta^\alpha(x) e^{-\lambda^\alpha t} \right| \leq C_0 e^{-(\lambda^\alpha + \gamma)t} \tag{18}$$

where $C_0 := (C + C' + CC') \|f\|_\infty$. Hence, for all $x \in \mathbb{N}$ and $\alpha \in \mathcal{A}_M$ such that $\lambda^\alpha > \beta$,

$$\left| J_\beta(x, \alpha) - \frac{\pi^\alpha(f^\alpha)}{\lambda^\alpha - \beta} \eta^\alpha(x) \right| \leq \int_0^{+\infty} e^{\beta t} \left| \mathbb{E}_x \left[f^\alpha \left(X_t^\alpha \right) \right] - \pi^\alpha(f^\alpha) \eta^\alpha(x) e^{-\lambda^\alpha t} \right| dt$$
$$\leq C_0 \int_0^{+\infty} e^{-(\lambda^\alpha + \gamma - \beta)t} \, dt \leq \frac{C_0}{\gamma}.$$

This ends the proof of the first assertion. As for the second one, we observe first that, under Hypotheses 4 and 5, there exists x_0 such that $\inf_{a \in A}\{f(x_0, a)\} > 0$. Together with Proposition 18, it yields for all $x \in \mathbb{N}$

$$\pi^\alpha(f^\alpha)\eta^\alpha(x) \geq \pi^\alpha(x_0)\eta^\alpha(x) \inf_{a \in A}\{f(x_0, a)\} > 0.$$

Further, it follows from (18) that for all $\beta \geq \lambda^\alpha$ and t sufficiently large,

$$e^{\beta t}\mathbb{E}_x\left[f^\alpha\left(X_t^\alpha\right)\right] \geq \pi^\alpha(f^\alpha)\eta^\alpha(x) - C_0 e^{-\gamma t}.$$

The conclusion follows immediatly by integration. $\qquad\square$

We are now in a position to prove Theorem 5. Recall that $\lambda^* = \sup_{\alpha \in \mathcal{A}_M} \lambda^\alpha$ is finite since λ^α is bounded from above by $\bar{d}_1$. Denote by $\mathcal{A}_M^*$ the collection of $\alpha \in \mathcal{A}_M$ such that $\lambda^\alpha = \lambda^*$.

Let us assume for the moment that $\mathcal{A}_M^*$ is non-empty. First, in view of Proposition 19, it holds for all $\beta < \lambda^*$ and $\alpha \in \mathcal{A}_M^*$,

$$\left|(\lambda^* - \beta)J_\beta(x, \alpha) - \pi^\alpha(f^\alpha)\eta^\alpha(x)\right| \leq C(\lambda^* - \beta).$$

It follows that

$$(\lambda^* - \beta)v_\beta(x) \leq \pi^\alpha(f^\alpha)\eta^\alpha(x) + C(\lambda^* - \beta).$$

Hence, we obtain

$$\limsup_{\beta \uparrow \lambda^*}(\lambda^* - \beta)v_\beta(x) \leq \inf_{\alpha \in \mathcal{A}_M^*}\{\pi^\alpha(f^\alpha)\eta^\alpha(x)\}. \tag{19}$$

Second, we denote by $\alpha_\beta \in \mathcal{A}_M$ an optimal Markov control for the β–discounted minimization problem as in Theorem 14. One clearly has

$$(\lambda^* - \beta)v_\beta(x) \geq (\lambda^{\alpha_\beta} - \beta)v_\beta(x).$$

To conclude the proof of Theorem 5, it is hence sufficient to show that $\mathcal{A}_M^* \neq \emptyset$ and that there exists $\alpha^* \in \mathcal{A}_M^*$ such that

$$\liminf_{\beta \uparrow \lambda^*}(\lambda^{\alpha_\beta} - \beta)v_\beta(x) = \pi^{\alpha^*}(f^{\alpha^*})\eta^{\alpha^*}(x). \tag{20}$$

Note that the existence of α^* such that $\lambda^{\alpha^*} = \lambda^*$ will validate the proof of (19) above.

Let us consider an increasing sequence $(\beta_n)_{n \in \mathbb{N}}$ converging to λ^* such that for all $x \in \mathbb{N}$,

$$\lim_{n \to +\infty}(\lambda^{\alpha_{\beta_n}} - \beta_n)v_{\beta_n}(x) = \liminf_{\beta \uparrow \lambda^*}(\lambda^{\alpha_\beta} - \beta)v_\beta(x).$$

Since A is compact, we can extract a subsequence (still denoted $(\beta_n)_{n \in \mathbb{N}}$) such that α_{β_n} converges pointwise to $\alpha^* \in \mathcal{A}_M$. The rest of the proof consists in showing that $\alpha^* \in \mathcal{A}_M^*$ and (20) is satisfied. To clarify the presentation, we split the proof of this result in three parts.

Step 1. As a first step, we show that $\alpha^* \in \mathcal{A}_M^*$ and the QSD $\pi^{\alpha_{\beta_n}}$ (resp. the extinction rate $\lambda^{\alpha_{\beta_n}}$) converges to π^{α^*} (resp. λ^{α^*}). In view of Lemma 20 below,

$(\pi^{\alpha_{\beta_n}})_{n\in\mathbb{N}}$ is a tight sequence of probability measures on $\mathbb{N}$. Hence, we can extract a subsequence converging pointwise to some probability measure π^* on $\mathbb{N}$. Denote by $\mathcal{K}^\alpha$ the adjoint operator of $\mathcal{L}^\alpha$, *i.e.*, for all $u \in \mathbb{R}^{\mathbb{Z}_+}$ and $x \in \mathbb{Z}_+$,

$$\mathcal{K}^\alpha u(x) = \sum_{y=0}^{x-1} b^\alpha(y) p^\alpha_{y,x-y} u(y) + d^\alpha(x+1)u(x+1) - (b^\alpha(x) + d^\alpha(x))\, u(x).$$

It follows from Proposition 4 of [MV12] that

$$\lambda^{\alpha_{\beta_n}} \pi^{\alpha_{\beta_n}} + \mathcal{K}^{\alpha_{\beta_n}} \pi^{\alpha_{\beta_n}} = 0. \tag{21}$$

In addition, Proposition 19 ensures that $\lambda^{\alpha_{\beta_n}} > \beta_n$ and thus $\lambda^{\alpha_{\beta_n}}$ converges to λ^*. Hence, by sending n to infinity, we obtain

$$\lambda^* \pi^* + \mathcal{K}^{\alpha^*} \pi^* = 0.$$

By Proposition 4 of [MV12], we deduce that π^* is a QSD of X^{α^*} with extinction rate λ^*. The conclusion follows by uniqueness of the QSD.

Step 2. As an intermediate step, we show that $\liminf_{\beta\uparrow\lambda^*} (\lambda^{\alpha_\beta} - \beta) v_\beta$ is collinear with η^{α^*}. Denote $\phi_n := (\lambda^{\alpha_{\beta_n}} - \beta_n)v_{\beta_n}$ and $\phi := \lim_{n\to+\infty} \phi_n$. It follows from Theorems 13 and 14 that

$$\beta_n \phi_n + \mathcal{L}^{\alpha_{\beta_n}} \phi_n + (\lambda^{\alpha_{\beta_n}} - \beta_n)\, f^{\alpha_{\beta_n}} = 0.$$

By sending n to infinity, we want to derive that

$$\lambda^* \phi + \mathcal{L}^{\alpha^*} \phi = 0. \tag{22}$$

In view of the continuity assumption of Hypothesis 4, it suffices to show that for all $x \in \mathbb{N}$,

$$\lim_{n\to+\infty} \sum_{k\geq 1} p^{\alpha_{\beta_n}}_{x,k} \phi_n(x+k) = \sum_{k\geq 1} p^{\alpha^*}_{x,k} \phi(x+k). \tag{23}$$

Let us show first that ϕ_n is bounded uniformly w.r.t. n. Applying Propositions 19 and 3, we obtain

$$\phi_n(x+k) \leq \pi^{\alpha_{\beta_n}}\left(f^{\alpha_{\beta_n}}\right) \eta^{\alpha_{\beta_n}}(x+k) + C\left(\lambda^{\alpha_{\beta_n}} - \beta_n\right)$$
$$\leq (C'+1)\,\|f\|_\infty + C\left(\lambda^* - \beta_n\right).$$

This implies that ϕ_n is uniformly bounded by $C_0 := (C'+1)\|f\|_\infty + C\left(\lambda^* - \beta_0\right)$. In addition, it holds

$$\left| \sum_{k\geq 1} p^{\alpha^*}_{x,k} \phi(x+k) - \sum_{k\geq 1} p^{\alpha_{\beta_n}}_{x,k} \phi_n(x+k) \right|$$
$$\leq C_0 \sum_{k\geq 1} \left| p^{\alpha^*}_{x,k} - p^{\alpha_{\beta_n}}_{x,k} \right| + \sum_{k\geq 1} p^{\alpha^*}_{x,k} |\phi(x+k) - \phi_n(x+k)|.$$

Using Scheffé's lemma and the dominated convergence theorem, we deduce that the relation (23) is satisfied, and hence (22). In other words, ϕ is a bounded

eigenfunction of $\mathcal{L}^{\alpha^*}$ for the eigenvalue $-\lambda^* = -\lambda^{\alpha^*}$. According to Corollary 2.4 of [CV15a], it yields that ϕ is collinear with η^{α^*}.

Step 3. To conclude, it remains to identify the collinearity coefficient between ϕ and η^{α^*}. It follows by integration of the HJB equation (11) that

$$\beta \pi^{\alpha_\beta} (v_\beta) + \pi^{\alpha_\beta} (\mathcal{L}^{\alpha_\beta} v_\beta) + \pi^{\alpha_\beta} (f^{\alpha_\beta}) = 0.$$

In particular, using the relation (21), we obtain

$$-\pi^{\alpha_{\beta_n}} (\phi_n) + \pi^{\alpha_{\beta_n}} (f^{\alpha_{\beta_n}}) = 0.$$

Now we can use Scheffé's Lemma and the dominated convergence theorem as above to derive

$$-\pi^{\alpha^*} (\phi) + \pi^{\alpha^*} \left(f^{\alpha^*} \right) = 0.$$

Since $\pi^{\alpha^*}(\eta^{\alpha^*}) = 1$ as stated in Proposition 3, we conclude that $\phi = \pi^{\alpha^*} \left(f^{\alpha^*} \right) \eta^{\alpha^*}$, and the proof of Theorem 5 is completed.

Lemma 20. *The family of probability measures* $(\pi^\alpha)_{\alpha \in \mathcal{A}_M}$ *is tight.*

Proof. It follows from Theorem 2 that for any $K \geq 1$,

$$\sum_{x \geq K} \pi^\alpha(x) \leq C \mathrm{e}^{-\gamma t} + \frac{\mathbb{P}_1 (X_t^\alpha \geq K)}{\mathbb{P}_1 (t < \tau^\alpha)}.$$

On the one hand, the Markov inequality and (4) yield

$$\mathbb{P}_1 (X_t^\alpha \geq K) \leq \frac{\mathrm{e}^{M\bar{b}t}}{K}.$$

On the other hand, as already mentioned, $\mathbb{P}_1(t < \tau^\alpha) \geq \mathrm{e}^{-\bar{d}_1 t}$. We deduce that, for all $t \geq 0$,

$$\sum_{x \geq K} \pi^\alpha(x) \leq C \mathrm{e}^{-\gamma t} + \frac{\mathrm{e}^{(M\bar{b} + \bar{d}_1)t}}{K}.$$

This ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6.2 Proof of Theorem 7

The proof of Theorem 7 follows from similar arguments as those developed in Section 6.1. In particular, it relies on the HJB equation and the existence of an optimal Markov control stated in Theorem 15. To apply this result, we need to show that w_β is bounded for $\beta < \lambda_* = \inf_{\alpha \in \mathcal{A}_M} \lambda^\alpha$. This is a delicate issue and we postpone its proof to Lemma 21 below.

Similar to the previous section, we start by assuming that there exists at least one control $\alpha \in \mathcal{A}_M$ such that $\lambda^\alpha = \lambda_*$. Using Proposition 19, one easily checks that

$$\liminf_{\beta \uparrow \lambda_*} (\lambda_* - \beta) \, w_\beta(x) \geq \sup_{\substack{\alpha \in \mathcal{A}_M \\ \lambda^\alpha = \lambda_*}} \{\pi^\alpha(f^\alpha)\eta^\alpha(x)\}.$$

To conclude the proof, it is hence enough to show that there exists $\alpha_* \in \mathcal{A}_M$ such that $\lambda^{\alpha_*} = \lambda_*$ and

$$\limsup_{\beta \uparrow \lambda_*} (\lambda_* - \beta) \, w_\beta(x) = \pi^{\alpha_*}(f^{\alpha_*})\eta^{\alpha_*}(x). \qquad\qquad (24)$$

For the sake of clarity, we split the proof in three steps.

First Step. Let us show first that there exists $\alpha \in \mathcal{A}_M$ such that $\lambda^\alpha = \lambda_*$. To achieve this, we want to extend the arguments of Step 1 in the proof of Theorem 5. Denote by α_β an optimal Markov control for the β–discounted maximization problem as in Theorem 15. In order to repeat Step 1, the only issue is to prove that — under appropriate assumptions on f — λ^{α_β} converges to λ_* as β goes to λ_*. Notice first that it follows from Propositions 19 and 3 that for all $\beta < \lambda_*$,

$$(\lambda^{\alpha_\beta} - \beta)w_\beta(x) \leq (C' + 1)\|f\|_\infty + C(\lambda^* - \beta). \tag{25}$$

Hence, we can conclude by showing that, for a specific function f, w_β tends to $+\infty$ as β goes to λ_*. Here we take $f(x, a) = 1$ for all $x \in \mathbb{N}$ and $a \in A$. In view of Proposition 19, we have for all $\beta < \lambda_*$,

$$J_\beta(x, \alpha) \geq \frac{1}{\lambda^\alpha - \beta} \inf_{\alpha \in \mathcal{A}_M} \{\eta^\alpha(x)\} - C,$$

where $\inf_{\alpha \in \mathcal{A}_M} \{\eta^\alpha(x)\} > 0$ by Proposition 18. By considering a family of $\alpha \in \mathcal{A}_M$ such that λ^α converges to λ_*, we deduce that

$$w_\beta(x) \geq \frac{1}{\lambda_* - \beta} \inf_{\alpha \in \mathcal{A}_M} \{\eta^\alpha(x)\} - C,$$

Hence, the conclusion follows immediately by sending β to λ_*.

Second Step. Let us show next that (24) holds true under the following assumption:
$$\sup_{\substack{\alpha \in \mathcal{A}_M \\ \lambda^\alpha = \lambda_*}} \{\pi^\alpha(f^\alpha)\} > 0.$$

Note that this condition is satisfied under Hypothesis 5. The idea is to extend the arguments of the previous section to prove that

$$\limsup_{\beta \uparrow \lambda_*} (\lambda^{\alpha_\beta} - \beta) w_\beta(x) = \pi^{\alpha_*}(f^{\alpha_*})\eta^{\alpha_*}(x),$$

where α_β is an optimal Markov control for the β–discounted maximization problem as in Theorem 15. In order to repeat Step 1 of Section 6.1, the only issue is to show that λ^{α_β} converges to λ_* as β goes to λ_*. To achieve this, we can extend the arguments above. Indeed, it follows from Proposition 19 that

$$w_\beta(x) \geq \frac{1}{\lambda_* - \beta} \sup_{\substack{\alpha \in \mathcal{A}_M \\ \lambda^\alpha = \lambda_*}} \{\pi^\alpha(f^\alpha)\eta^\alpha(x)\} - C.$$

We deduce that w_β tends to $+\infty$ as β goes to λ_*. Together with (25), this imposes that λ^{α_β} tends to λ_*. In order to repeat Step 2 of Section 6.1, we have to ensure that
$$\limsup_{\beta \uparrow \lambda_*} (\lambda^{\alpha_\beta} - \beta) w_\beta(x) < +\infty.$$

This is a straightforward consequence of (25). The rest of the proof follows easily by repeating the arguments of Section 6.1.

Third Step. Let us show finally that (24) holds true under the following assumption:

$$\sup_{\substack{\alpha \in \mathcal{A}_M \\ \lambda^\alpha = \lambda_*}} \{\pi^\alpha(f^\alpha)\} = 0.$$

We argue by considering two cases. If λ^{α_β} converge to λ_*, we can once again repeat the arguments of the previous section to reach the conclusion. If λ^{α_β} does not converges to λ_*, then it follows from (25) and the monotonicity of $\beta \to w_\beta(x)$ that for all $\beta < \lambda_*$,

$$w_\beta(x) \le \lim_{\beta \uparrow \lambda_*} w_\beta(x) < +\infty.$$

Hence, (24) holds true once again.

Lemma 21. *For every $\beta < \lambda_*$, the value function w_β is bounded.*

Proof. If we knew that the optimal control in the value function is Markov, this would directly follow from Proposition 19. The difficulty is to prove that non-Markov controls cannot give unbounded costs. Denote

$$\beta_* := \inf \left\{ \beta; \ \sup_{x \in \mathbb{N}} w_\beta(x) = +\infty \right\}.$$

We assume that $\beta_* < \lambda_*$ in order to reach a contradiction.

First step. We start by showing that w_{β_*} is bounded. For all $\beta < \beta_*$, it follows from Theorem 15 that the optimal control is Markov, and hence from Propositions 3 and 19 that

$$w_\beta(x) = J_\beta(x, \alpha_\beta) \le \frac{\pi^{\alpha_\beta}(f^{\alpha_\beta})}{\lambda^{\alpha_\beta} - \beta} \eta^{\alpha_\beta}(x) + C \le \frac{(C'+1)\|f\|_\infty}{\lambda_* - \beta} + C.$$

We deduce that

$$w_{\beta_*}(x) = \lim_{\beta \uparrow \beta_*} w_\beta(x) \le \frac{(C'+1)\|f\|_\infty}{\lambda_* - \beta_*} + C < +\infty.$$

Note that the first identity above relies on a permutation of supremum since $\beta \mapsto w_\beta(x)$ is nondecreasing for all $x \in \mathbb{N}$.

Second step. To reach a contradiction, it remains to prove that there exists $\varepsilon > 0$ such that

$$\sup_{x \in \mathbb{N}} w_{\beta_* + \varepsilon}(x) < +\infty.$$

Without loss of generality we take $f = \mathbb{1}_\mathbb{N}$ and we claim that for all $\alpha \in \mathcal{A}$ and $x \in \mathbb{N}$,

$$\int_0^{+\infty} s^n e^{\beta_* s} \mathbb{P}_x(s < \tau^\alpha) \, ds \le C_0^{n+1} n!, \tag{26}$$

where

$$C_0 := \sup_{x \in \mathbb{N}} \sup_{\alpha \in \mathcal{A}} \left\{ \int_0^{+\infty} e^{\beta_* s} \mathbb{P}_x(s < \tau^\alpha) \, ds \right\}.$$

If (26) is proved, the conclusion follows from the fact that, for any $\varepsilon < \frac{1}{C_0}$,

$$w_{\beta_* + \varepsilon}(x) = \sup_{\alpha \in \mathcal{A}} \left\{ \int_0^{+\infty} e^{(\beta_* + \varepsilon)s} \mathbb{P}_x(s < \tau^\alpha) \, ds \right\} \le C_0 \sum_{n \ge 0} (C_0 \varepsilon)^n = \frac{C_0}{1 - \varepsilon C_0}.$$

Let us show that the relation (26) holds by induction. The result is obvious for $n = 0$. Using successively Fubini-Tonelli and the pseudo-Markov property (Lemma 10), the incrementation step follows by

$$\int_0^{+\infty} s^{n+1} e^{\beta_* s} \mathbb{P}_x \left(s < \tau^\alpha \right) ds$$

$$= \int_0^{+\infty} \int_t^{+\infty} s^n e^{\beta_* s} \mathbb{P}_x \left(s < \tau^\alpha \right) ds\, dt$$

$$= \int_0^{+\infty} \int_0^{+\infty} (t+s)^n e^{\beta_*(t+s)} \mathbb{E}_x \left[\mathbb{P}_{X_t^\alpha} \left(s < \tau^{\alpha^t} \right) \right] ds\, dt$$

$$= \sum_{k=0}^n \binom{n}{k} \int_0^{+\infty} t^k e^{\beta_* t} \int_0^{+\infty} s^{n-k} e^{\beta_* s} \mathbb{E}_x \left[\mathbb{P}_{X_t^\alpha} \left(s < \tau^{\alpha^t} \right) \right] ds\, dt$$

$$= \sum_{k=0}^n \binom{n}{k} \int_0^{+\infty} t^k e^{\beta_* t} \sum_{y \in \mathbb{N}} \mathbb{E}_x \left[\mathbb{1}_{X_t^\alpha = y} \int_0^{+\infty} s^{n-k} e^{\beta_* s} \mathbb{P}_y \left(s < \tau^{\alpha^t} \right) ds \right] dt$$

It follows that

$$C_{n+1} \le \sum_{k=0}^n \binom{n}{k} C_k C_{n-k},$$

where

$$C_k := \sup_{y \in \mathbb{N}} \sup_{\alpha \in \mathcal{A}} \int_0^{+\infty} s^k e^{\beta_* s} \mathbb{P}_y \left(s < \tau^\alpha \right) ds.$$

To conclude, it remains to observe that the sequence $u_n := C_0^{n+1} n!$ is the solution of

$$u_{n+1} = \sum_{k=0}^n \binom{n}{k} u_k u_{n-k}$$

satisfying $u_0 = C_0$. $\qquad\square$

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