Degenerate processes killed at the boundary of a domain

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Abstract

We investigate certain properties of degenerate Feller processes that are killed when exiting a relatively compact set. Our main result provides general conditions ensuring that such a process possesses a (possibly non unique) quasi stationary distribution. Conditions ensuring uniqueness and exponential convergence are discussed. The results are applied to stochastic differential equations.

Keywords: Absorbed Markov processes, Quasi-Stationary distributions, Exponential mixing, Hypoellipticity, Stochastic differential equations.

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1 Introduction

The purpose of this paper is to investigate certain properties of (possibly degenerate) Feller processes that are killed when exiting a relatively compact set.

To give the flavor of the kind of results that will be proved here, consider a stochastic differential equation on $M = \mathbb{R}^n$,

$$dX_t = S^0(X_t)dt + \sum_{j=1}^{m} S^j(X_t) \circ dB_t^j, \tag{1}$$

where $\circ$ refers to the Stratonovich stochastic integral, $S^0, S^j, j = 1, \ldots, m$ are smooth vector fields on M and $B^1, \ldots, B^m$ m independent Brownian motions. As usual, the law of $(X_t)_{t \geq 0}$ when $X_0 = x$ is denoted $\mathbb{P}_x$. For $x, y \in M$ and $A \subset M$, we write $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$, $\|x\| = \sqrt{\langle x, x \rangle}$ and $d(x, A) = \inf_{y \in A} \|x - y\|$.

Let $D \subset M$ be an open connected set with compact closure $K = \overline{D}$ and boundary $\partial D = K \backslash D$. We assume that ∂D is *regular* (or, according to a usual terminology, *satisfies the exterior sphere condition*) in the following sense. For all $p \in \partial D$, there exists a (possibly non unique) unit vector v (i.e $\|v\| = 1$) and $r > 0$ such that

$$d(p + rv, K) = r.$$

Such a v is called a *unit outward normal vector* at p.

Remark 1 Let $p \in \partial D$ and $U \subset \mathbb{R}^n$ be an open neighborhood of p. If $\partial D \cap U$ is C^2, there is a unique unit outward normal vector at p. If $D \cap U$ is convex, there is a (possibly non unique) unit outward normal vector at p. If $D \subset \mathbb{R}^2$ is a nonconvex polygon and $p \in \partial D$ is a vertex at which the interior angle is $> \pi$, there is no unit outward normal at p.

Associated to (1) is the Stroock and Varadhan deterministic control system:

$$\dot{y}(t) = S^0(y(t)) + \sum_{j=1}^{m} u^j(t) S^j(y(t)) \tag{2}$$

where the control function $u = (u^1, \ldots, u^m) : \mathbb{R}_+ \mapsto \mathbb{R}^m$, can be chosen to be piecewise continuous. Given such a control function, we let $y(u, x, \cdot)$ denote the maximal solution to (2) starting from x (i.e $y(u, x, 0) = x$).

Let $x \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$be an open set. We say that U is *accessible* from x if there exists a control u and $t \geq 0$ such that $y(u, x, t) \in U$. If in addition, $y(u, x, s) \in D$ for all $0 \leq s \leq t$, (in which case $x \in D$ and $U \cap D \neq \emptyset$) we say that U is *D-accessible* from x. By abuse of language, we say that a point y is accessible (respectively D-accessible) from x provided every neighborhood U of y is accessible (respectively D-accessible) from x.

We let

$$\tau_D^{out} = \inf\{t \geq 0 : \ X_t \in M \setminus D\}.$$

Theorem 1 *Consider the following conditions:*

(i) *The* outside *set $M \setminus K$ is accessible from all $x \in K$;*

(ii) (Boundary conditions)

- **(a)** *For every $p \in \partial D$, there exists an outward unit normal vector v at p, such that*

$$\sum_{i=1}^{m} \langle S^i(p), v \rangle^2 \neq 0;$$

- **(b)** *For some $\varepsilon > 0$, the set $D_\varepsilon = \{x \in D : \ d(x, \partial D) > \varepsilon\}$ is D-accessible from all $x \in D \setminus D_\varepsilon$.*

Then, under condition (i), $\mathbb{P}_x(\tau_D^{out} < \infty)$ for all $x \in D$ and under conditions $(i), (ii)$, there exists a probability measure μ on D such that

$$\mathbb{P}_\mu(X_t \in \cdot | \tau_D^{out} > t) = \mu(\cdot). \tag{3}$$

A probability measure μ satisfying (3) is called a *Quasi-Stationary Distribution* (QSD) [16].

Remark 2 Simple criteria ensuring conditions (i) and $(ii) - (b)$ in Theorem 1 will be discussed in Section 3 (Propositions 14 and 16). In particular, Proposition 16 has the following useful consequence:

If ∂D is C^2, then condition $(ii) - (a)$ implies condition $(ii) - (b)$.

If the SDE (1) enjoys certain *hypoellipticity* properties, more can be said. Given a family $\mathcal{S}$ of smooth vector fields on M and $k \in \mathbb{N}$, we let $[\mathcal{S}]_k$ denote the set of vector fields recursively defined by $[\mathcal{S}]_0 = \mathcal{S}$, and

$$[\mathcal{S}]_{k+1} = [\mathcal{S}]_k \cup \{[Y, Z] \ : Y, Z \in [\mathcal{S}]_k\}$$

where $[Y, Z]$ stands for the Lie bracket of Y and Z. Set $[\mathcal{S}] = \cup_k [\mathcal{S}]_k$ and $[\mathcal{S}](x) = \{Y(x) \ : Y \in [\mathcal{S}]\}$.

We shall say here that point $x^* \in M$ satisfies the *weak Hörmander condition* (respectively the *Hörmander condition*, respectively the *strong Hörmander condition*) if $[\{S^0, \ldots, S^m\}](x^*)$ (respectively

$$\{S^1(x^*), \ldots, S^m(x^*)\} \cup \{[Y, Z](x^*) : \ Y, Z \in [\{S^0, \ldots, S^m\}]\},$$

respectively

$$[\{S^1, \ldots, S^m\}](x^*)$$

spans $\mathbb{R}^n$.

Theorem 2 *Assume that:*

(i) *Conditions* $(i), (ii) - (a)$ *of Theorem 1 hold;*

(ii) *The weak Hörmander condition is satisfied at every point* $x \in K$;

(iii) *Every* $y \in D$ *is* D*-accessible from every* $x \in D$.

Then the QSD μ *is unique. Its topological support equals* K *and it has a smooth density with respect to the Lebesgue measure.*

Suppose furthermore that there exists a point $x^* \in D$ *at which the weak Hörmander condition is strengthened to the Hörmander condition. Then there exist* $\alpha > 0$, $C \in (0, +\infty)$ *and a continuous function* $h : D \mapsto]0, \infty[$ *with* $h(x) \to 0$, *as* $x \to \partial D$, *satisfying* $\mu(h) = 1$, *such that for all* $\rho \in \mathcal{M}_1(D)$ *(the set of probability measures over* D*),*

$$\|\mathbb{P}_\rho(X_t \in \cdot | \tau_D^{out} > t) - \mu(\cdot)\|_{TV} \leq \frac{C}{\rho(h)} e^{-\alpha t}$$

where $\| \cdot \|_{TV}$ *stands for the total variation distance.*

Corollary 3 *Assume that the condition* $(ii) - (a)$ *of Theorem 1 holds, and that the strong Hörmander condition is satisfied at every point* $x \in K$. *Then, the conclusions of Theorem 2 holds and the density of* μ *is positive on* D.

Proof: Let, for $\varepsilon \geq 0$, $y^\varepsilon(x, u, \cdot)$ be defined like $y(x, u, \cdot)$ when S^0 is replaced by εS^0. By Chow's theorem [5] (see also [18]), the strong Hörmander condition implies that for all $x, y \in D$ there exists a control u piecewise continuous with $u^i(s) \in \{-1, 0, 1\}$ and $t \geq 0$ such that $y^0(x, u, s) \in D$ for all $0 \leq s \leq t$ and $y^0(x, u, t) = y$. By continuity (or a simple application of Gronwall's lemma), $y^\varepsilon(x, u, \cdot) \to y^0(x, u, \cdot)$ uniformly on $[0, t]$ as $\varepsilon \to 0$. Let $u^\varepsilon(s) = \frac{u(\frac{s}{\varepsilon})}{\varepsilon}$. Then $y^\varepsilon(x, u, s) = y(x, u^\varepsilon, \varepsilon s)$. This proves that y is D-accessible from x. Because the strong Hörmander condition also holds in a neighborhood of K, the same proof also shows that for y in a neighborhood of K and $x \in K$, y is accessible from x. The conditions, hence the conclusions, of Theorem 1 are then satisfied. Positivity of $\frac{d\mu}{dx}$ is proved in Lemma 22. $\square$

Remark 3 The results above extend easily to the situation where M is a n-dimensional manifold, provided ∂D is a C^2 sub-manifold.

Remark 4 The recent paper [9] considers QSDs and their properties under the assumption that the underlying process is strong-Feller. Quasi-stationarity for degenerate strong-Feller diffusion processes have also been the focus of interest in the recent literature ([9, 15, 20]). Observe that none of the conditions of Theorem 1, neither the weak Hörmander condition assumed in Theorem 2 imply that the SDE (1) is strong Feller. However, we will see that our assumptions imply that the SDE is Feller.

Example 1 Suppose M is the cylinder, $M = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$. Let $m = 1$,

$$S^0(x, y) = \partial_x \text{ and } S^1(x, y) = \partial_y.$$

Let $D = \mathbb{R}/\mathbb{Z} \times]0, 1[$. Here the conditions of Theorem 2, are easily seen to be satisfied. However, the process is not strong Feller (the dynamics in the x-variable being deterministic). It is not hard to check that the unique QSD is the measure

$$\mu(dxdy) = 2\frac{\sin(\pi y)}{\pi}\mathbf{1}_D(x, y)dxdy. \tag{4}$$

Example 2 (Example 1, continued) Let M and D be like in Example 1, $m = 1$,

$$S^0(x, y) = a(y)\partial_x \text{ and } S^1(x, y) = \partial_y,$$

where a is a smooth function ≥ 1. Like in example 1, the unique QSD μ is given by (4). Suppose that $a'(y^*) \neq 0$ for some $0 < y^* < 1$. Then the Hörmander condition holds at (x, y^*), so that by Theorem 2, $(\mathbb{P}_x(X_t \in \cdot | \tau_D^{out} > t))_{t \geq 0}$ converges at an exponential rate to μ.

Remark 5 In absence of condition (iii) in Theorem 2, there is no guarantee that the QSD is unique. Still every QSD has a smooth density (see Lemma 22).

Example 3 Let $M = \mathbb{R}$ and $D =]0,5[$. We consider smooth functions $\varphi^1, \varphi^2, \psi^1, \psi^2 : D \to [0,1]$ such that $\psi^1 + \psi^2 = 1$ and

$$\begin{cases} \varphi^1_{|]0,1]} \equiv 1, \ \varphi^1_{|[1,2]} \le 1, \ \varphi^1_{|[2,5[} \equiv 0, \\ \varphi^2_{|]0,3]} \equiv 0, \ \varphi^2_{|[3,4]} \le 1, \ \varphi^2_{|[4,5[} \equiv 1, \\ \psi^1_{|]0,2]} \equiv 1, \ 0 < \psi^1_{|]2,3[} \le 1, \ \psi^1_{|[3,5[} \equiv 0, \\ \psi^2_{|]0,2]} \equiv 0, \ 0 < \psi^2_{|]2,3[} \le 1, \ \psi^2_{|[3,5[} \equiv 1. \end{cases}$$

For all $\alpha > 0$, we consider the absorbed diffusion process X^α evolving according to the Itô SDE

$$dX^\alpha_t = \left(\varphi^1(X^\alpha_t) + \sqrt{\alpha}\, \varphi^2(X^\alpha_t)\right) dB_t + (\psi^1(X_t) + \alpha\, \psi^2(X_t))\, dt.$$

and absorbed when it reaches $\partial D = \{0,5\}$. While X^α satisfies the conditions (i) and (ii) of Theorem 2, it does not satisfy condition (iii) and it admits either one or two QSDs, depending on the value of α. Indeed, as shown in [3], there exists $\alpha_c > 0$ such that

- for all $\alpha \in]0, \alpha_c]$, X^α admits a unique QSD supported by $[3,5]$,
- for all $\alpha \in]\alpha_c, +\infty[$, X^α admits exactly two QSDs, supported respectively by $[3,5]$ and $[0,5]$.

Outline The rest of the paper is organized as follows. Section 2 sets up the notation and proves the main results: a general existence result (Theorem 7), a uniqueness criterion (Theorem 10) and a convergence theorem (Theorem 12). These results are used in Section 3 to prove the Theorems 1 and 2 stated in the introduction.

2 Killed processes

Throughout, we let M denote a separable and locally compact metric space, and $D \subset M$ a nonempty set with compact closure $K = \overline{D}$, such that D is *relatively open* in K. That is $D = \mathcal{O} \cap K$ for some open set $\mathcal{O} \subset M$.

We let $(P_t)_{t \ge 0}$ denote a Markov Feller semi-group on M. By this, we mean (as usual) that $(P_t)_{t \ge 0}$ is a semi-group of Markov operators on $C_0(M)$ (the space of continuous functions on M vanishing at infinity) and that $P_t f(x) \to f(x)$ as $t \to 0$ for all $f \in C_0(M)$. Observe that since we are interested by the behavior of the process killed outside K, the behavior of $(P_t)_{t \ge 0}$ at infinity is irrelevant (and the reader can think of M as compact without loss of generality).

By classical results (see e.g Le Gall [14], Theorem 6.15), there exist a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ with $(\mathcal{F}_t)$ right continuous and complete, a family of probabilities $(\mathbb{P}_x)_{x \in M}$ on $(\Omega, \mathcal{F})$ and a continuous time adapted process (X_t) on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ taking values in M, such that:

(i) (X_t) has cad-lag paths,

(ii) $\mathbb{P}_x(X_0 = x) = 1$ and,

(iii) (X_t) is a Markov process with semigroup (P_t), meaning that

$$\mathbb{E}_x(f(X_{t+s})|\mathcal{F}_t) = P_s f(X_t)$$

for all $t, s \geq 0$ and f measurable bounded (or ≥ 0).

For any Borel set $A \subset M$ we let $\tau_A = \inf\{t \geq 0 : \ X_t \in A\}$ and $\tau_A^{out} = \tau_{M \setminus A}$. The assumptions on $(\mathcal{F}_t)$ (right continuous and complete) imply that τ_A and τ_A^{out} are stopping times with respect to $(\mathcal{F}_t)$ (see e.g Bass [1]).

Remark 6 For $X_0 = x \in D$, $\tau_D^{out} \leq \tau_K^{out}$ but it is not true in general that $\tau_D^{out} = \tau_K^{out}$. Consider for example the ode on $\mathbb{R}^2$ given by

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 0 \end{cases}$$

Let

$$D = \{(x, y) \in \mathbb{R}^2 \ -1 < x < 1, -1 < y < x^2\}.$$

For $-1 < x < 0$ and $y = 0$ the trajectory $(x(t), y(t)) = (x + t, 0)$ starting at $(x, 0)$ leaves D at time $-x$ and K at time $-x + 1$.

An open set $U \subset M$ is said *accessible* from $x \in M$ if there exists $t \geq 0$ such that $P_t(x, U) = P_t \mathbf{1}_U(x) > 0$. A point y is said *accessible* from $x \in M$ if every open neighborhood U of y is accessible from x.

We shall assume throughout the following standing assumption.

Hypothesis 1 *For all $x \in K, M \setminus K$ is accessible from x.*

Proposition 4 *There exist positive constants C, Λ such that*

$$\mathbb{P}_x(\tau_K^{out} > t) \leq Ce^{-\Lambda t}$$

for all $x \in K$. In particular $\tau_K^{out} < \infty$, and hence $\tau_D^{out} < \infty$, $\mathbb{P}_x$ almost surely for all $x \in K$.

Proof: By Feller continuity and Portmanteau theorem, for all $t \geq 0$, the set $O_t = \{x \in M : P_t(x, U) > 0\}$ is open (possibly empty). By assumption, the family $\{O_t : t \in \mathbb{R}^+\}$ covers K, so that, by compactness, there exist $t_1, \ldots, t_n$ such that $K \subset \cup_{i=1}^n O_{t_i}$. In particular, for some $\delta > 0$ and $t = \max\{t_1, \ldots, t_n\}$ $\mathbb{P}_x(\tau_U > t) \leq (1-\delta)$ for all $x \in K$. Thus, by the Markov property, $\mathbb{P}_x(\tau_U > kt) \leq (1-\delta)^k$. This proves the result. □

2.1 Green kernel and QSDs

Let $B(D)$ denote the set of bounded measurable functions $f : D \mapsto \mathbb{R}$. For all $f \in B(D), x \in D$ and $t \geq 0$ set

$$P_t^D f(x) = \mathbb{E}_x(f(X_t)\mathbf{1}_{\tau_D^{out} > t}).$$

By Proposition 4, $(P_t^D)_{t \geq 0}$ is a well defined sub-Markovian semigroup on $B(D)$. The semigroup property is a consequence of the Markov property and the fact that τ_D^{out} is a stopping time.

The *Green kernel* G^D is the bounded (by Proposition 4) operator defined on $B(D)$ by

$$G^D f(x) = \int_0^\infty P_t^D f(x) dt = \mathbb{E}_x(\int_0^{\tau_D^{out}} f(X_t) dt).$$

For all $x \in D$ and $A \subset M$, a Borel set, we let

$$G^D(x, A) = G^D \mathbf{1}_{A \cap D}(x).$$

Quasi-stationary Distributions

A *Quasi-stationary Distribution* (QSD) for (P_t^D) is a probability measure μ on D such that

$$\mu P_t^D = e^{-\lambda t} \mu \tag{5}$$

for some $\lambda > 0$. For further reference we call λ the *absorption rate* (or simply the rate) of μ. Equivalently,

$$\frac{\mu P_t^D(\cdot)}{\mu P_t^D \mathbf{1}_D} = \mathbb{P}_\mu(X_t \in \cdot \mid \tau_D^{out} > t) = \mu(\cdot).$$

Lemma 5 *Equation (5) holds if and only if* $\mu G^D = \frac{1}{\lambda}\mu$.

Proof: Clearly, by definition of G^D, equation (5) implies that $\mu G^D = \frac{1}{\lambda}\mu$. Conversely, assume that $\mu G^D = \frac{1}{\lambda}\mu$. Then for every bounded non-negative measurable map $f : D \mapsto \mathbb{R}$, $\mu(G^D f) = \frac{1}{\lambda}\mu(f)$ and also

$$\mu(G^D P_t^D f) = \frac{1}{\lambda}\mu(P_t^D f).$$

That is

$$\mu(\int_t^\infty P_s^D f) = \frac{1}{\lambda}\mu(P_t^D f).$$

Equivalently,

$$\mu(G^D f - \int_0^t P_s^D f) = \frac{1}{\lambda}\mu(P_t^D f).$$

This shows that the map $v(t) = \mu(P_t^D f)$ satisfies the integral equation

$$v(t) - v(0) = -\lambda \int_0^t v(s) ds.$$

It follows that $v(t) = v(0)e^{-\lambda t}$. □

Let $C_b(D) \subset B(D)$ denote the set of bounded continuous functions on D, and $C_0(D) \subset C_b(D)$ the subset of functions f such that $f(x) \to 0$ when $x \to \partial_K D := K \setminus D$.

Remark 7 In the recent paper [8], the authors prove the existence and convergence to a QSD under the condition that the sub-Markovian semigroup is strong-Feller. Observe that in our case, although (P_t) is Feller, there is no evidence in general that (P_t^D) is strong-Feller nor that it preserves $C_b(D)$. On the other hand, under rather weak, reasonable conditions, G^D maps $C_b(D)$ into $C_0(D)$, as illustrated by the following example.

Example 4 Consider the ode on $\mathbb{R}$ given by $\dot{x} = -1$. For $D =]0, 1[$ $P_t^D f(x) = f(x-t)\mathbf{1}_{x>t}$ is not Feller, $G^D f(x) = \int_0^x f(u)du$ is Feller (and even strong Feller). If now $D =]0, 1]$, then G^D maps $C_b(D)$ into $C_0(D)$.

The condition that $G^D(C_b(D)) \subset C_0(D)$ plays a key role in the next Theorem 7 and will be investigated in the subsequent sections.

An open set $U \subset M$ is said *D-accessible* from $x \in D$ if $P_t^D(x, U) > 0$ for some $t \geq 0$. Point $y \in K$ is said *D-accessible* from $x \in D$ if every open neighborhood of y is D-accessible from x.

Lemma 6 *An open set $U \subset M$ is D-accessible from $x \in D$ if and only if $G^D(x,U) > 0$. In particular, the set of D-accessible points from x coincide with the topological support of $G^D(x,\cdot)$.*

Proof: Suppose that $P_t^D(x,U) > 0$ for some open set U and $t \geq 0$. Fatou Lemma and right continuity of paths imply that

$$\liminf_{s\to t, s>t} P_s^D(x,U) \geq \mathbb{E}_x(\liminf_{s\to t, s>t} \mathbf{1}_U(X_s)\mathbf{1}_{\tau_D^{out}>s)} \geq P_t^D(x,U) > 0.$$

This proves that $s \to P_s^D(x,U)$ is positive on some interval $[t, t+\varepsilon]$, hence $G^D(x,U) > 0$. The converse implication is obvious. □

Theorem 7 *Assume that:*

(i) $G^D(C_b(D)) \subset C_0(D)$;

(ii) *There exists an open set $U \subset M$, D-accessible from all $x \in D$ and such that $\overline{U \cap K} \subset D$.*

Then there exists a QSD for (P_t^D).

Remark 8 A sufficient condition ensuring condition (ii) in Theorem 7 is that there exists a point $y \in D$ which is D-accessible from every $x \in D$.

Proof: We claim that there exists a function $\phi \in C_0(D)$, strictly positive on D and $\theta > 0$ such that

$$G^D\phi \geq \theta\phi. \tag{6}$$

Proof of the claim: For $\varepsilon > 0$, let

$$\overline{U}_K^\varepsilon = \{x \in K : \, d(x, U \cap K) \leq \varepsilon\}.$$

Choose $\varepsilon > 0$ small enough so that $\overline{U}_K^\varepsilon \subset D$ (recall that D is relatively open in K). Let $\psi(x) = (1 - \frac{d(x,U\cap K)}{\varepsilon})^+$. For all $x \in D$, $G^D\psi(x) \geq G^D\mathbf{1}_{U\cap K}(x) = G^D\mathbf{1}_U(x) > 0$. Thus, by compactness of $\overline{U}_K^\varepsilon$, $\theta = \inf_{x\in\overline{U}_K^\varepsilon} G^D\psi(x) > 0$. Since $\psi = 0$ on $K \setminus \overline{U}_K^\varepsilon$, it follows that for all $x \in D, G^D\psi(x) \geq \theta\psi(x)$. The map $\phi = G^D\psi$ is positive and satisfies (6).

Let $B(K)$ (respectively $C(K)$) be the space of bounded (respectively continuous) functions over K. The operator G^D extends to a bounded operator G on $B(K)$ defined as

$$Gf(x) = \begin{cases} G^D(f|_D)(x) & \text{for } x \in D \\ 0 & \text{for } x \in \partial_K D \end{cases}$$

By condition (i), G leaves $C(K)$ invariant. That is $G(C(K)) \subset C(K)$.

Let $\mathcal{M}_1(\phi)$ be the set of Borel finite nonnegative measures μ on K such that $\mu(\phi) = 1$ and let $T : \mathcal{M}_1(\phi) \mapsto \mathcal{M}_1(\phi)$ be the map defined by

$$T(\mu) = \frac{\mu G}{\mu G \phi}$$

We first observe that T is continuous for the weak* topology: if $\mu_n \rightharpoonup \mu$ for the weak* topology in $\mathcal{M}_1(\phi)$, then $\mu_n G \rightharpoonup \mu G$ and $\mu_n G\phi \to \mu G\phi$. Since in addition $\mu_n G\phi \geq \theta \mu_n(\phi) = \theta > 0$ by (6), $T(\mu_n) \rightharpoonup T(\mu)$.

Choose an open neighborhood $\mathcal{N}$ of $\partial_K D$ such that $G\mathbf{1}_K \leq \theta/2$ on $\mathcal{N} \cap K$ and set $C = \sup_{x \in K \setminus \mathcal{N}} \frac{(G\mathbf{1}_K)(x)}{\phi(x)} < \infty$. Then, for all $\mu \in \mathcal{M}_1(\phi)$

$$T(\mu)(K) = \frac{\mu(G\mathbf{1}_K)}{\mu G\phi} \leq \frac{\mu(G\mathbf{1}_K)}{\theta} \leq \frac{\theta/2\mu(K) + C\mu(\phi)}{\theta} = \frac{\mu(K)}{2} + \frac{C}{\theta}.$$

It follows that for any $R \geq 2\frac{C}{\theta}$ the set

$$\mathcal{M}_1^R(\phi) = \{\mu \in \mathcal{M}_1(\phi) \ : \mu(K) \leq R\}$$

is invariant by T. Since $\mathcal{M}_1^R(\phi)$ is convex and compact (for the weak* topology), T admits a fixed point by Tychonoff Theorem. If μ is such a fixed point, the probability $\frac{\mu(\cdot \cap D)}{\mu(D)}$ is then a QSD for (P_t^D). □

Remark 9 The proof of Theorem 7 is reminiscent of the proof of the existence Theorem 4.2 in [6].

2.2 Uniqueness and convergence criteria

Right eigenfunctions

Suppose that the conditions, hence the conclusion, of Theorem 7 hold. We say that $h \in C_0(D)$ is a *positive right eigenfunction* for G^D if $h(x) > 0$ for all $x \in D$ and

$$G^D h = \frac{1}{\lambda} h \tag{7}$$

for some $\lambda > 0$.

Lemma 8 *If h is a positive right eigenfunction, the parameter λ in (7) necessarily equals the absorption rate of any QSD and*

$$P_t^D h = e^{-\lambda t} h$$

for all $t \geq 0$.

Proof: If μ is a QSD with rate λ', then $\mu G^D h = \frac{1}{\lambda'}\mu h = \frac{1}{\lambda}\mu h$ so that $\lambda = \lambda'$. The proof of the second statement is similar to the proof of Lemma 5 and left to the reader. □

The following result shows that a strengthening in the assumptions of Theorem 7 ensures the existence of a positive right eigenfunction for G^D.

Corollary 9 *Assume that:*

(i) $G^D(C_b(D)) \subset C_0(D)$ *and* G^D *is a compact operator on* $C_0(D)$*;*

(ii) *For all* $x, y \in D$ y *is* D*-accessible from* x*.*

Then, there exists a positive right eigenfunction for G^D*.*

Proof: Let $r = \lim_{n\to\infty} \|(G^D)^n\|^{1/n}$ be the spectral radius of G^D on $C_0(D)$. Let μ be a QSD (whose existence is given by Theorem 7) with rate λ. For any $f \in C_0(D)$ such that $0 \le f \le 1$ and $\mu(f) \neq 0$ $\|(G^D)^n\| \ge \mu((G^D)^n f) = \frac{1}{\lambda^n}\mu(f)$. Hence $r \ge \frac{1}{\lambda} > 0$.

Let $C_0^+(D) = \{f \in C_0(D) : f \ge 0\}$. It is readily seen that $C_0^+(D)$ is a *reproducing* cone in $C_0(D)$, invariant by G^D, meaning that $C_0(D)^+$ is a cone, $C_0(D) = \{u - v : u, v \in C_0(D)^+\}$ and $G^D(C_0(D)^+) \subset C_0(D)^+$. Therefore, by Krein Rutman Theorem, there exists $h \in C_0(D)^+ \setminus \{0\}$ such that $Gh = rh$. Let $y \in D$ be such that $h(y) > 0$. Then, $h \ge \frac{h(y)}{2}\mathbf{1}_{U\cap K}$ for some neighborhood U of y. Therefore, by (ii),

$$rh(x) = (G^D h)(x) \ge \frac{h(y)}{2}(G^D \mathbf{1}_{U\cap K})(x) > 0.$$

Now $\frac{1}{\lambda}\mu(h) = \mu(G^D h) = r\mu(h)$. Hence $r = \frac{1}{\lambda}$. This concludes the proof. □

Uniqueness and convergence

We say that (P_t^D) is *irreducible* if there exists a nontrivial measure ξ on D such that for all $x \in D$ and A Borel,

$$\xi(A) > 0 \Rightarrow G^D(x, A) > 0.$$

Theorem 10 *Assume that:*

(i) *The hypotheses of Theorem 7 hold;*

(ii) *There exists a positive right eigenfunction for* G^D*;*

(iii) *(P_t^D) is irreducible.*

Then (P_t^D) has a unique QSD.

Remark 10 Conditions (i) and (ii) of Theorem 10 are implied by the assumptions of Corollary 9.

Proof: Let h be a positive right eigenfunction and μ a QSD with rate λ. Let Q and π respectively denote the Markov kernel and the probability on D defined by

$$Q(f) = \lambda \frac{G^D(fh)}{h},$$

and

$$\pi(f) = \frac{\mu(fh)}{\mu(h)}$$

for all $f \in B(D)$. Then, π is invariant by Q. The assumption that (P_t^D) is irreducible makes Q irreducible, in the sense that $\xi(A) > 0 \Rightarrow Q(x, A) > 0$ for all $x \in D$ and A Borel. Therefore, by a standard result (see e.g [7] or [17]), π is the unique invariant probability of Q. Assume now that ν is another QSD with rate α. Then $\nu(Gh) = \frac{1}{\alpha}\nu(h) = \frac{1}{\lambda}\nu(h)$. Hence $\alpha = \lambda$. It follows that the probability π' defined like π with ν in place of μ is invariant by Q. By uniqueness, $\pi = \pi'$ and consequently $\mu = \nu$. □

A sufficient (and often more tractable than the definition) condition ensuring irreducibility is given by the next lemma.

Lemma 11 *Suppose that there exists an open set $U \subset M$, D-accessible from all $x \in D$, and a non trivial measure ξ such that for all $x \in U$ $G^D(x, \cdot) \geq \xi(\cdot)$. Then (P_t^D) is irreducible.*

Proof: For all $x \in D$, there exists, by D-accessibility, $t \geq 0$ such that $P_t^D(x, U) > 0$. For every Borel set $A \subset M$,

$$\begin{aligned} G^D(x, A) &\geq \int_0^\infty P_{t+s}^D(x, A)ds = \int_0^\infty \int_M P_t^D(x, dy)P_s^D(y, A)ds \\ &\geq \int_0^\infty \int_U P_t^D(x, dy)P_s^D(y, A)ds \geq P_t(x, U)\xi(A). \end{aligned}$$

□

If the local minorization $G^D(x, \cdot) \geq \xi(\cdot)$ appearing in Lemma 11 can be improved to local minorization involving (P_t^D), we also get the exponential convergence of the conditional semigroup toward μ. More precisely,

Theorem 12 *Suppose that:*

(i) *The hypotheses of Theorem 7 hold;*

(ii) *There exists a positive right eigenfunction for G^D, denoted by h;*

(iii) *There exist an open set $U \subset M$, D-accessible from all $x \in D$, a non trivial measure ξ with $\xi(U) > 0$, and positive numbers $T > \varepsilon > 0$ such that for all $x \in U, T - \varepsilon \leq t \leq T$,*

$$P_t^D(x, \cdot) \geq \xi(\cdot).$$

Then there exist $C, \alpha > 0$ such that, for all $\rho \in \mathcal{M}_1(D)$ (the set of probability measures over D),

$$\left\| \frac{\rho P_t^D}{\rho P_t^D \mathbf{1}_D} - \mu(\cdot) \right\|_{TV} \leq \frac{C\mu(h)}{\rho(h)} e^{-\alpha t}$$

where $\| \cdot \|_{TV}$ stands for the total variation distance.

Proof: Let h be a positive right eigenfunction and μ a QSD with rate λ. For all $f \in B(D)$ and $t \geq 0$, let

$$Q_t(f) = e^{\lambda t} \frac{P_t^D(fh)}{h}$$

and

$$\pi(f) = \frac{\mu(fh)}{\mu(h)}$$

It is easy to verify that (Q_t) is a Markov semigroup (usually called the Q process induced by μ) having π as invariant probability. In order to prove the theorem we will show that :

Step 1 There exists a probability ν on D such that for every compact set $\tilde{K} \subset D$ there is some integer n and some constant c (both depending on $\tilde{K}$) such that

$$Q_{nT}(x, \cdot) \geq c\nu(\cdot)$$

for all $x \in \tilde{K}$.

Step 2 The function $V = \frac{1}{h}$ is a continuous and proper *Lyapunov function* for Q_T, that is

$$\lim_{x \to \partial D} V(x) = \infty,$$

and

$$Q_T V \leq \rho V + C \tag{8}$$

for some $0 \leq \rho < 1$ and $C \geq 0$.

From step 2 we deduce that for all $n \geq 0$

$$Q_{nT}V \leq \rho^n V + \frac{C}{1-\rho}. \tag{9}$$

Choose $R > \frac{2C}{1-\rho}$ and set $\tilde{K} = \{x \in D : V(x) \leq R\}$. Then $\tilde{K}$ is a compact subset of D and, by step 1, there is some $n \geq 1$ such that

$$Q_{nT}(x, \cdot) \geq c\nu(.) \tag{10}$$

on $\tilde{K}$. Now, relying on a version of Harris's theorem proved by Hairer and Mattingly in [10], (9) and (10) imply that for all $f : D \mapsto \mathbb{R}$ measurable, and $k \geq 0$,

$$|Q^k_{nT}(f)(x) - \pi(f)| \leq cste\ \gamma^k(1 + V(x))\|f\|_V$$

for all $x \in D$, where $0 \leq \gamma < 1$ and $\|f\|_V = \sup_{x\in D} \frac{|f(x)|}{1+V(x)}$. Recalling that h is bounded (it is continuous on the compact K), this last inequality entails that there exist $C, \alpha > 0$ such that, for any $t \geq 0$ and f such that $\|f\|_{1/h} \leq 1$,

$$|Q_t f(x) - \pi(f)| \leq \frac{C}{h(x)} e^{-\alpha t}.$$

In particular, for any f such that $\|f\|_\infty \leq 1$ and $\rho \in \mathcal{M}_1(D)$ such that $\rho(1/h) < +\infty$,

$$\left|\rho Q_t \left[\frac{f}{h}\right] - \frac{\mu(f)}{\mu(h)}\right| \leq C\rho(1/h)e^{-\alpha t}.$$

For any $\rho \in \mathcal{M}_1(D)$, denote $h \circ \rho(dx) := \frac{h(x)\rho(dx)}{\rho(h)}$. Then, for any f such that $\|f\|_\infty \leq 1$ and $\rho \in \mathcal{M}_1(D)$,

$$\frac{\mu(f)}{\mu(h)} - \frac{C}{\rho(h)}e^{-\alpha t} \leq (h \circ \rho)Q_t \left[\frac{f}{h}\right] \leq \frac{\mu(f)}{\mu(h)} + \frac{C}{\rho(h)}e^{-\alpha t}.$$

Moreover, since $P^D_t f(x) = e^{-\lambda t}h(x)Q_t[f/h](x)$, then

$$\rho P^D_t f = e^{-\lambda t}\rho(h) \times (h \circ \rho)Q_t[f/h].$$

Thus, fixing $\rho \in \mathcal{M}_1(D)$ and denoting $t_\rho := \frac{1}{\alpha} \log\left(\frac{C\mu(h)}{\rho(h)}\right)$, for any $t > t_\rho$,

$$\frac{\frac{\mu(f)}{\mu(h)} - \frac{C}{\rho(h)}e^{-\alpha t}}{\frac{1}{\mu(h)} + \frac{C}{\rho(h)}e^{-\alpha t}} \leq \frac{\rho P^D_t f}{\rho P^D_t 1} \leq \frac{\frac{\mu(f)}{\mu(h)} + \frac{C}{\rho(h)}e^{-\alpha t}}{\frac{1}{\mu(h)} - \frac{C}{\rho(h)}e^{-\alpha t}}.$$

Computations entail that, setting $C(\rho) := \frac{2\mu(h)}{1-e^{-\alpha}} \times \frac{C}{\rho(h)}$, for any $t \geq 1 + t_\rho$,

$$\frac{\frac{\mu(f)}{\mu(h)} + \frac{C}{\rho(h)} e^{-\alpha t}}{\frac{1}{\mu(h)} - \frac{C}{\rho(h)} e^{-\alpha t}} \leq \mu(f) + C(\rho) e^{-\alpha t}.$$

More directly, one has also

$$\frac{\frac{\mu(f)}{\mu(h)} - \frac{C}{\rho(h)} e^{-\alpha t}}{\frac{1}{\mu(h)} + \frac{C}{\rho(h)} e^{-\alpha t}} \geq \mu(f) - 2\frac{C\mu(h)}{\rho(h)} e^{-\alpha t}.$$

Hence, one shows that there exists $C' > 0$ such that, for any probability measure ρ and any $t > 1 + t_\rho$,

$$\left\| \frac{\rho P_t^D}{\rho P_t^D 1} - \mu \right\|_{TV} \leq \frac{C' \mu(h)}{\rho(h)} e^{-\alpha t}.$$

For $t \leq 1 + t_\rho$,

$$\left\| \frac{\rho P_t^D}{\rho P_t^D 1} - \mu \right\|_{TV} \leq 2 \leq 2e^{\alpha(1+t_\rho)} e^{-\alpha t} = \frac{2e^{\alpha} C \mu(h)}{\rho(h)} e^{-\alpha t}.$$

To sum up, there exists two constants $C, \alpha > 0$ such that, for any $\rho \in \mathcal{M}_1(D)$ and $t \geq 0$,

$$\left\| \frac{\rho P_t^D}{\rho P_t^D 1} - \mu \right\|_{TV} \leq \frac{C \mu(h)}{\rho(h)} e^{-\alpha t}.$$

This concludes the proof.

We now pass to the proof of steps 1 and 2.

Proof of Step 1: For $\delta > 0$ let $O_\delta = \{x \in D : \ G^D(x, U) > \delta\}$. It is not hard to see that O_δ is open in D. Indeed, suppose to the contrary that there exists a sequence $x_n \to x \in D$ with

$$G^D(x_n, U) \leq \delta < G^D(x, U).$$

Let $\psi_\varepsilon(y) = \min(1, \frac{d(y, M\setminus U)}{\varepsilon})$. By continuity of ψ_ε and condition (i) of Theorem 7,

$$G^D \psi_\varepsilon(x) = \lim_{n\to\infty} G^D \psi_\varepsilon(x_n) \leq \limsup_{n\to\infty} G^D \mathbf{1}_U(x_n) \leq \delta.$$

On the other hand, by monotone convergence

$$\lim_{\varepsilon\to 0} G^D \psi_\varepsilon(x) = G^D \mathbf{1}_U(x) > \delta.$$

A contradiction.

By D-accessibility the family $(O_\delta)_{\delta>0}$ covers D. Thus, for every compact set $\tilde{K} \subset D$ there exists $\delta > 0$ such that for all $x \in \tilde{K}, G^D(x,U) \geq \delta$. Now, relying on Proposition 4 and the definition of G^D, one can choose $S > 0$ large enough so that

$$\int_0^S P_t^D(x,U)dt > \frac{\delta}{2}$$

for all $x \in \tilde{K}$. Consequently, for all $x \in \tilde{K}$ there is some $0 \leq t_x \leq S$ such that

$$P_{t_x}^D(x,U) \geq \frac{\delta}{2S} = \delta'.$$

Hence,

$$Q_{t_x}(x,U) \geq \delta''$$

for some $\delta'' > 0$ and all $x \in \tilde{K}$. By the assumption on (P_t^D) there exists $c' > 0$ such that

$$Q_t(x,\cdot) \geq c'\xi'(\cdot)$$

for all $x \in U$ and $T - \varepsilon \leq t \leq T$, where $\xi'(f) = \xi(fh)$. Choose now n sufficiently large so that $\frac{S}{n} < \varepsilon$. Then for all $x \in \tilde{K}, nT = t_x + n\tau_x$ for some $\tau_x \in [T-\varepsilon, T]$. Thus

$$Q_{nT}(x,\cdot) \geq \int_U Q_{t_x}(x,dy)Q_{n\tau_x}(y,\cdot) \geq \delta''(c'\xi'(U))^{n-1}c'\xi'(\cdot).$$

Proof of Step 2: Without loss of generality, we may assume (to shorten notation) that $T = 1$.

By Markov inequality,

$$\mathbb{P}_x(\tau_D^{out} > 1) \leq \mathbb{E}_x(\tau_D) = G^D\mathbf{1}_D(x).$$

Let $\theta > 0$ be such that $\rho = e^\lambda\theta < 1$. By the assumption that $G^D(C_b(D)) \subset C_0(D)$, there exists $\varepsilon > 0$ and $C' \geq 0$ such that

$$\mathbb{P}_x(\tau_D^{out} > 1) \leq \theta + C'\mathbf{1}_{\{x\in D:\, d(x,\partial D)\geq\varepsilon\}}.$$

Then

$$Q_1(V)(x) = e^\lambda\frac{\mathbb{P}_x(\tau_D^{out} > 1)}{h(x)} \leq \rho V(x) + C$$

for some $C \geq 0$. □

3 Application to SDEs

The main purpose of this section is to prove Theorems 1 and 2. We assume throughout this section that $M = \mathbb{R}^n$ and $D \subset M$ is an open connected set with compact closure K. We let (P_t), respectively (P_t^D), denote the Markov, respectively sub-Markov, semigroup induced by the SDE (1).

3.1 Accessibility

This section discusses the accessibility properties and provides simple criteria ensuring that conditions (i) (accessibility of $M \setminus K$ from K) and $(ii) - (b)$ (accessibility of D_ε from D) of Theorem 1 are satisfied.

Recall that $y(u, x, \cdot)$ denotes the maximal solution to the control system (2) starting from x (i.e $y(u, x, 0) = x$).

The following proposition easily follows from the celebrated Strook and Varadhan's support theorem [23] (see also Theorem 8.1, Chapter VI in [13]). It justifies the terminology used in Section 1.

Proposition 13 *Assume that the S^j are bounded with first and second bounded derivatives. Let $x \in M$ and $U \subset M$ open. Then U is accessible (respectively D-accessible) from x if there exists a control u and $t \geq 0$ such that $y(u, x, t) \in U$ (respectively $y(u, x, t) \in U$ and $y(u, x, s) \in D$ for all $0 \leq s \leq t$).*

Remark 11 The assumption that the S^j are bounded with first and second bounded derivatives is free of charge here, since - by compactness of K - we can always modify the S^j outside K so that they have compact support.

This proposition provides a tool to prove that condition (i) of Theorem 1 (or the standing Hypothesis 1) holds true. As an illustration, we provide a proof of the following result, originally due to Pinsky [19].

Proposition 14 *Suppose there exists $\tilde{x} \in M \setminus K$ and $\delta > 0$ such that for all $x \in D$*

$$\sum_{j=1}^{m} \langle S^j(x), x - \tilde{x} \rangle^2 \geq \delta \|x - \tilde{x}\|^2. \tag{11}$$

Then $M \setminus K$ is accessible from all $x \in K$.

Proof: Note that by continuity and compactness (of $K \times \{x \in \mathbb{R}^n : \|x\| = 1\}$), one can always assume that (11) holds true on some larger open bounded domain D' with $K \subset D'$. For $x \in D$ and $j \in \{1, \ldots, m\}$ set

$$u^j(x) = -\frac{1}{2\delta\varepsilon}\langle S^j(x), x - \tilde{x}\rangle$$

where ε will be chosen later. Consider the deterministic ode

$$\dot{x} = S^0(x) + \sum_j u^j(x) S^j(x).$$

Set $v(t) = \|x(t) - \tilde{x}\|^2$. Then, as long as $x(t) \in D'$,

$$\frac{dv(t)}{dt} \leq -\frac{1}{\varepsilon}v(t) + a$$

where $a = \sup_{x \in D'} 2(\langle S^0(x), x - \tilde{x}\rangle)$. Thus

$$v(t) := \|x(t) - \tilde{x}\|^2 \leq e^{-\varepsilon t}(v_0 - a\varepsilon) + a\varepsilon.$$

One can choose ε small enough so that $x(t)$ meets $D' \setminus K$. □

The next result (Proposition 16) provides a natural condition ensuring that condition $(ii) - (b)$ of Theorem 1 holds.

Suppose that ∂D is regular, as defined in the introduction. Let $\mathcal{N}_p$ denote the set of unit outward normal vectors at $p \in \partial D$ and let

$$\mathcal{N} = \{(p, v) : \ p \in \partial D, v \in \mathcal{N}_p\}.$$

We say that a vector $w \in \mathbb{R}^n$ *points inward* D at $p \in \partial D$ if

$$\langle w, v\rangle \leq 0 \text{ for all } v \in \mathcal{N}_p, \ \textbf{and} \ \langle w, v\rangle < 0 \text{ for at least one } v \in \mathcal{N}_p.$$

We say that it *points strictly inward* D at p if $\langle w, v\rangle < 0$ **for all** $v \in \mathcal{N}_p$. We say that a vector field F points inward (respectively strictly inward) D if $F(p)$ points inward (strictly inward) D at p for all $p \in \partial D$.

Lemma 15 *Suppose ∂D is regular. Let F be a Lipschitz vector field pointing inward D and let $\Phi = \{\Phi_t\}$ be its flow. Then*

(i) $\Phi_t(K) \subset D$ *for all* $t > 0$;

(ii) *There exists a compact set $A \subset D$ invariant under $\{\Phi_t\}$ (i.e $\Phi_t(A) = A$ for all $t \in \mathbb{R}$) such that for all $x \in D$ $\omega_\Phi(x) \subset A$, where $\omega_\Phi(x)$ stands for the omega limit set of x for Φ.*

Proof: We first show that $\Phi_t(K) \subset K$ for all $t \geq 0$. Suppose not. Then, for some $p \in \partial D$ and $\varepsilon > 0$, $d(\Phi_t(p), K) > 0$ on $]0, \varepsilon]$. The function

$$t \to V(t) := d(\Phi_t(p), K),$$

being Lipschitz on $[0, \varepsilon]$ it is absolutely continuous, hence almost everywhere derivable and $V(t) = \int_0^t \dot{V}(u)du$. Let $0 < t_0 \leq \varepsilon$ be a point at which it is derivable, $x_0 = \Phi_{t_0}(p)$ and $p_0 \in \partial D$ be such that $\|x_0 - p_0\| = d(x_0, K)$. Then for all $t > 0$,

$$\frac{V(t_0 + t) - V(t_0)}{t} = \frac{d(\Phi_t(x_0), K) - d(x_0, K)}{t} \leq \frac{\|\Phi_t(x_0) - p_0\| - \|x_0 - p_0\|}{t}.$$

Letting $t \to 0$, we get

$$\dot{V}(t_0) \leq \frac{\langle F(x_0), x_0 - p_0 \rangle}{\|x_0 - p_0\|} \leq \frac{\langle F(x_0) - F(p_0), x_0 - p_0 \rangle}{\|x_0 - p_0\|}$$
$$\leq L\|x_0 - p_0\| = LV(t_0),$$

where L is a Lipschitz constant for F and the second inequality comes from the fact that $F(p_0)$ points inward D at p_0. By Gronwall's lemma we then get that for all $0 < s < t \leq \varepsilon$, $V(t) \leq e^{L(t-s)}V(s)$. Since $V(0) = 0$, V cannot be positive. This proves the desired result. Note that this first result doesn't require that ∂D is regular but only that for all $p \in \partial D$ at which $\mathcal{N}_p \neq \emptyset$ $\langle F(p), v \rangle \leq 0$ for all $v \in \mathcal{N}_p$ (compare with Theorem 2.3 in [4]).

We now show that $\Phi_t(K) \subset D$ for $t > 0$. This amounts to show that for all $p \in \partial D$ and all $\varepsilon > 0$ small enough, $\Phi_{-\varepsilon}(p) \in M \setminus K$. Let $p \in \partial D$. Choose $v \in \mathcal{N}_p$ such that $\delta := -\langle F(p), v \rangle > 0$. By assumption $d(p+rv, K) = r$ for some $r > 0$. Thus for all $\varepsilon > 0$ such that $\varepsilon\|F(p)\|^2 < r\delta$,

$$\|(p - \varepsilon F(p)) - (p + rv)\|^2 < r^2 - \varepsilon\delta r.$$

Since $\|\Phi_{-\varepsilon}(p) - (p - \varepsilon F(p)\| = o(\varepsilon)$, this shows that $\Phi_{-\varepsilon}(p) \notin K$ for $\varepsilon > 0$ small enough.

Assertion (ii) is a consequence of (i). It suffices to set

$$A = \cap_{t \geq 0} \Phi_t(K).$$

□

For $(p, v) \in \mathcal{N}$, set $R(p, v) = \sup\{r > 0 \ : d(p + rv, K) = r\} \in]0, \infty]$, and let

$$R_{\partial D} = \inf_{(p,v) \in \mathcal{N}} R(p, v).$$

It will be assumed in Proposition 16 that $R_{\partial D} \neq 0$.

Remark 12 If D is convex or ∂D is C^2, then $R_{\partial D} \neq 0$. Note however that the assumption that ∂D is regular doesn't imply in general that $R_{\partial D} \neq 0$. Here is a simple example. For $1 \leq \alpha \leq 2$, Let $D^\alpha \subset \mathbb{R}^2$ be defined as

$$D^\alpha = \{(x,y) \in \mathbb{R}^2 \ : 0 < x < 1, |y| < x^\alpha\}.$$

It is not hard to verify that ∂D^α is regular for all $1 \leq \alpha \leq 2$. However, if $1 < \alpha < 2$, then $\mathcal{N}_{(0,0)} = \{v \ : \|v\| = 1, v_1 < 0\}$ and

$$\lim_{v \to (0,1), v \in \mathcal{N}_{(0,0)}} R((0,0),v) = 0.$$

Proposition 16 *Assume that ∂D is regular and $R_{\partial D} \neq 0$.*

(i) *For each $p \in \partial D$, the two following conditions, denoted $(a),(b)$, are equivalent:*

(a) *$\mathcal{N}_p \cap -\mathcal{N}_p = \emptyset$ and for each $v \in \mathcal{N}_p$*

$$\langle S^0(p), v\rangle < 0 \text{ or } \sum_{i=1}^m \langle S^i(p), v\rangle^2 \neq 0;$$

(b) *There exists a vector $w \in \mathsf{Span}\{S^1(p),\ldots,S^m(p)\}$ such that $S^0(p) + w$ points strictly inward D at p.*

(ii) *If for all $p \in \partial D$ condition $(i)-(a)$ (or $(i)-(b)$) holds, then*

$$D_\varepsilon = \{x \in D : \ d(x, \partial D) > \varepsilon\}$$

is accessible from all $x \in D$ for some $\varepsilon > 0$.

Proof: Point (ii). We first assume that $(i)-(b)$ holds at every $p \in \partial D$ and prove that D_ε is accessible for some $\varepsilon > 0$. The assumption $R_{\partial D} > 0$ makes $\mathcal{N}$ closed (hence compact). Indeed, if $(p_n, v_n) \to (p, v)$ with $(p_n, v_n) \in \mathcal{N}$. Then, for any $0 < r < R_{\partial D}$, $d(p_n + rv_n, K) = r$. Thus $d(p + rv, K) = r$.

This has the consequence that, if a continuous vector field F points strictly inward D at $p \in \partial D$, it points strictly inward D at $q \in \partial D$ for all q in a neighborhood of p. Therefore, by compactness, there exists a covering of ∂D by open sets $U_1,\ldots,U_k$, and vector fields $W_1,\ldots,W_k \in \mathsf{Span}\{S^1,\ldots,S^m\}$ such that for all $p \in \partial D \cap U_i$ $F_i(p) := S^0(p) + W_i(p)$ points strictly inward D at p. Set $U_0 = M \setminus \partial D$ and let $\{\rho_i\}_{i=0,\ldots,k}$ be a partition of unity subordinate to $\{U_i\}_{i=0,\ldots,k}$. That is $\rho_i \in C^\infty(M), \rho_i \geq$

$0, \sum_{i=0}^k \rho_i = 1$, and $supp(\rho_i) \subset U_i$. Define $F = \rho_0 S^0 + \sum_{i=1}^k \rho_i F_i$. Then, F points strictly inward D and writes

$$F = S^0 + \sum_{i=1}^{m} u^i S^i \tag{12}$$

with $u^i \in C^\infty(M)$. In view of (12), Proposition 13 and Lemma 15, this proves the result.

Point (i). We now prove that conditions $(i)-(a)$ and $(i)-(b)$ are equivalent. The implication $(i)-(b) \Rightarrow (i)-(a)$ is straightforward. We focus on the converse implication. Let

$$Cone(\mathcal{N}_p) = \{tv, t \geq 0, v \in \mathcal{N}_p\}$$

and $conv(\mathcal{N}_p)$ be the convex hull of $\mathcal{N}_p$. We claim that

$$conv(\mathcal{N}_p) \subset Cone(\mathcal{N}_p)$$

and $0 \notin conv(\mathcal{N}_p)$. To prove the first inclusion, it suffices to show that $Cone(\mathcal{N}_p)$ is convex. To shorten notation, assume (without loss of generality) that $p = 0$. Let $x, y \in Cone(\mathcal{N}_0)$ and $0 \leq t \leq 1$. By definition of $\mathcal{N}_0$, $Cone(\mathcal{N}_0) = \{z \in \mathbb{R}^n, \exists r > 0 \text{ s.t. } d(rz, K) = \|rz\|\}$, so there exists $r > 0$ such that $d(rx, K) = \|rx\|$ and $d(ry, K) = \|ry\|$. Thus for all $z \in K$

$$\|rx - z\|^2 - \|rx\|^2 = \|z\|^2 - 2\langle rx, z\rangle \geq 0.$$

Similarly $\|z\|^2 - 2\langle ry, z\rangle \geq 0$. Thus

$$\|r(tx + (1-t)y) - z\|^2 - \|r(tx + (1-t)y)\|^2$$

$$= t(\|z\|^2 - 2\langle rx, z\rangle) + (1-t)(\|z\|^2 - 2\langle ry, z\rangle) \geq 0.$$

This proves that $tx + (1-t)y \in Cone(\mathcal{N}_0)$, hence convexity of $Cone(\mathcal{N}_0)$.

The fact that $0 \notin conv(\mathcal{N}_p)$ follows from the assumption that $\mathcal{N}_p \cap -\mathcal{N}_p = \emptyset$. Indeed, suppose to the contrary that $0 = \sum_{i=1}^k t_i x_i$ with $k \geq 2, x_i \in \mathcal{N}_p, t_i > 0$ and $\sum_{i=1}^k t_i = 1$. Then

$$-\frac{t_1}{1-t_1} x_1 \in conv(x_2, \ldots, x_k) \subset conv(\mathcal{N}_p) \subset cone(\mathcal{N}_p).$$

Thus $-x_1 \in \mathcal{N}_p$. A contradiction.

We shall now deduce the implication $(i)-(a) \Rightarrow (i)-(b)$ from the Minimax theorem (see e.g [22]). For all $j \in \{1, \dots, m\}$ set $S^{-j} = -S^j$. Let $J = \{-m, \dots, 0, \dots m\}$ and

$$\Delta(J) = \{\alpha \in \mathbb{R}^J : \alpha_j \geq 0, \sum_{j \in J} \alpha_j = 1\}.$$

By condition $(i)-(a)$ and compactness of $\mathcal{N}_p$ there exists $\delta > 0$ such that for all $v \in \mathcal{N}_p$

$$\min_{j \in J} \langle S^j(p), v \rangle \leq -\delta.$$

Thus for all $v \in Cone(\mathcal{N}_p)$

$$\min_{j \in J} \langle S^j(p), v \rangle \leq -\delta \|v\|.$$

The set $\mathcal{N}_p$ being compact (in finite dimension) its convex hull is also compact by Carathéodory theorem. Thus, because $0 \notin conv(\mathcal{N}_p)$, $\|v\| \geq \frac{\delta'}{\delta}$ for some $\delta' > 0$ and all $v \in conv(\mathcal{N}_p)$. It then follows that

$$\sup_{v \in conv(\mathcal{N}_p)} \inf_{\alpha \in \Delta(J)} \langle \sum_{j \in J} \alpha_j S^j(p), v \rangle \leq \sup_{v \in conv(\mathcal{N}_p)} \min_{j \in J} \langle S^j(p), v \rangle \leq -\delta'$$

for some $\delta' > 0$. By the Minimax theorem, the left hand side, also equals

$$\inf_{\alpha \in \Delta(J)} \sup_{v \in conv(\mathcal{N}_p)} \langle \sum_{j \in J} \alpha_j S^j(p), v \rangle$$

and this infimum is achieved for some $\beta \in \Delta(J)$. If $\beta_0 \neq 0$ this implies that

$$\sup_{v \in conv(\mathcal{N}_p)} \langle S^0(p) + \sum_{j \in J, j \neq 0} \frac{\beta_j}{\beta_0} S^j(p), v \rangle \leq -\frac{\delta'}{\beta_0} < 0.$$

If $\beta_0 = 0$, for $R > 0$ sufficiently large

$$\sup_{v \in conv(\mathcal{N}_p)} \langle S^0(p) + R \sum_{j \in J} \beta_j S^j(p), v \rangle \leq -R\delta' + \|S^0(p)\| < 0.$$

This concludes the proof. □

Remark 13 It follows from Proposition 16 that whenever ∂D is C^2, condition $(ii)-(a)$ of Theorem 1 implies condition $(ii)-(b)$, because at each point $p \in \partial D$ there is a unique outward unit normal. The following example shows that this is not true in general. Let D^1 be as in Remark 12, with $\alpha = 1$, and let (X_t) be solution to

$$dX_t = S^1(X_t) \circ dB_t$$

where $S^1(x, y) = (1, 2)$. At each point $p \in \partial D$ there is at least one $v \in \mathcal{N}_p$ such that $\langle S^1(p), v\rangle \neq 0$ so that condition $(ii) - (a)$ is satisfied. However, for $0 < \eta < \varepsilon, D^1_\varepsilon$ is not D-accessible from $(\eta, 0)$.

Observe also that none of the conditions required in Proposition 16 is necessary for D_ε to be accessible. Let D^α be as in Remark 12, with $1 < \alpha \leq 2$ and let

$$dX_t = e_1 \circ dB^1_t + e_2 \circ dB^2_t,$$

with (e_1, e_2) the canonical basis of $\mathbb{R}^2$. As shown in Corollary 3 (and its proof), D^α_ε is accessible; while for $1 < \alpha < 2, R_{\partial D} = 0$ and for $\alpha = 2, \mathcal{N}_{0,0} \cap -\mathcal{N}_{0,0} \neq \emptyset$.

3.2 Proof of Theorem 1

In order to prove Theorem 1, it suffices to show that the two assumptions of Theorem 7 are satisfied. The second assumption is implied by condition $(ii) - (b)$ of Theorem 1. We shall show here that the first one is implied by conditions (i) and $(ii) - (a)$ of Theorem 1.

Feller properties

Let $C(\mathbb{R}_+, M)$ be the set of continuous paths $\eta : \mathbb{R}_+ \mapsto M$ equipped with the topology of uniform convergence on compact intervals.

Let $(\eta_n)_{n\geq 0}$ be a sequence converging to η in $C(\mathbb{R}_+, M)$. Set

$$\tau^{n,out}_D = \inf\{t \geq 0 : \eta_n(t) \in M \setminus D\}, \tau^{out}_D = \inf\{t \geq 0 : \eta(t) \in M \setminus D\}$$

Define $\tau^{n,out}_K$ and τ^{out}_K similarly.

Lemma 17 **(i)** *Suppose $\eta(0) \in D$. Then for all $f \in C_b(D), f \geq 0$ and all $t \geq 0$,*

$$\liminf_{n\to\infty} f(\eta_n(t))\mathbf{1}_{\{\tau^{n,out}_D > t\}} \geq f(\eta(t))\mathbf{1}_{\{\tau^{out}_D > t\}}.$$

In particular

$$\liminf_{n\to\infty} \tau^{n,out}_D \geq \tau^{out}_D.$$

(ii) *Suppose $\eta(0) \in K$. Then*

$$\limsup_{n\to\infty} \tau_K^{n,out} \leq \tau_K^{out}$$

and for all $f \in C_0(K), f \geq 0$ and all $t \geq 0$,

$$\limsup_{n\to\infty} f(\eta_n(t))\mathbf{1}_{\{\tau_K^{n,out}>t\}} \leq f(\eta(t))\mathbf{1}_{\{\tau_K^{out}>t\}}.$$

Proof: (i) If $\tau_D^{out} \leq t$ the statement is obvious. If $\tau_D^{out} > t$, then $\eta([0,t]) \subset D$ so that, for n large enough, $\eta_n([0,t]) \subset D$. That is $\tau_D^{n,out} > t$ and the statement follows. The assertion that $\liminf_{n\to\infty} \tau_D^{n,out} \geq \tau_D^{out}$ follows by choosing $f = \mathbf{1}_D$.

(ii) Suppose to the contrary that $\tau_K^{n,out} > \tau_K^{out}+\varepsilon$ for some $\varepsilon > 0$ and infinitely many n. Then $\eta^n([0,\tau_K^{out}+\varepsilon]) \subset K$. Hence $\eta([0,\tau_K^{out}+\varepsilon]) \subset K$. A contradiction. The last assertion directly follows for $t \neq \tau_K^{out}$ and $f \in C_b(K)$. If now $t = \tau_K^{out}$ and $f \in C_0(K)$, $f(\eta^n(t)) \to f(\eta(t)) = 0$. □

In the following lemma, P_t^K denotes the semigroup defined as $P_t^K f(x) = \mathbb{E}_x(f(X_t)\mathbf{1}_{\tau_K^{out}>t})$.

Lemma 18 *Suppose that condition (i) of Theorem 1 holds. Then,*

(i) *For all $f \geq 0, f \in C_b(D), x \in D$ and $t \geq 0$,*

$$\liminf_{y\to x} P_t^D f(y) \geq P_t^D f(x),$$

(ii) *For all $f \geq 0, f \in C_0(K), x \in K$ and $t \geq 0$*

$$\limsup_{y\to x, y\in K} P_t^K f(y) \leq P_t^K f(x)$$

(iii) *Suppose, in addition, that condition $(ii)-(a)$ of Theorem 1 holds. Then,*

(a) *For all $x \in \partial D$,*

$$\mathbb{P}_x(\tau_K^{out} = 0) = 1,$$

and for all $x \in D$,

$$\mathbb{P}_x(\tau_D^{out} = \tau_K^{out}) = 1;$$

(b) *For all $f \in C_0(D)$ and $t \geq 0$, $P_t^D(f) \in C_b(D)$;*

(c) *For all $f \in C_b(D)$ and $t \geq 0$, $G^D f \in C_0(D)$. Condition (i) of Theorem 7 is then satisfied.*

Proof: Let (X_t^x) be the (strong) solution to (1) with initial condition $X_0^x = x$. We can always assume that $(X_t^x)_{t\geq 0,\, x\in M}$ is defined on the Wiener space space $C(\mathbb{R}_+, \mathbb{R}^m)$ equipped with its Borel sigma field and the Wiener measure $\mathbb{P}$ (the law of $(B^1, \ldots, B^m)$). That is $\mathbb{P}_x(\cdot) = \mathbb{P}(X^x \in \cdot)$. Also, for all $\omega \in C(\mathbb{R}_+, \mathbb{R}^m)$, the map $x \in M \to X^x(\omega) \in C(\mathbb{R}_+, M)$ is continuous (see for instance [14], Theorem 8.5).

Assertions (i) and (ii) then follow from Lemma 17 and Fatou's lemma.

We now pass to the proof of (iii).

a) Let $p \in \partial D$. By condition $(ii) - (a)$ of Theorem 1, there exist a unit vector v and $r > 0$ such that the map $\Psi : \mathbb{R}^n \mapsto \mathbb{R}$ defined as

$$\Psi(x) = r^2 - \|x - (p + rv)\|^2,$$

satisfies

$$\Psi(x) > 0 \Rightarrow x \in M \setminus K,$$

and $\langle \nabla\Psi(p), S^j(p)\rangle \neq 0$ for some $j \in \{1, \ldots, m\}$. Without loss of generality we can assume that $j = 1$. Hence, for some neighborhood U of p,

$$(\langle \nabla\Psi(x), S^1(x)\rangle)^2 \geq a > 0$$

on $\overline{U}$.

By Ito's formulae

$$\Psi(X^p_{t\wedge\tau_U^{out}}) = \int_0^{t\wedge\tau_U^{out}} L\Psi(X_s^p)ds + M_{t\wedge\tau_U^{out}}$$

where

$$L\Psi = S^0(\Psi) + \frac{1}{2}\sum_{j=1}^m (S^j)^2(\Psi),$$

$$M_t = \sum_{j=1}^m \int_0^t \sigma^j(X_s^p)dB_s^j,$$

and $\sigma^j(x)$ is any bounded measurable function coinciding with $\langle \nabla\Psi(x), S^j(x)\rangle$ on $\overline{U}$. In the definition of L above, we used the standard notation for differential operators defined from vector fields: $S^i(f)(x) = \langle S^i(x), \nabla f(x)\rangle$. For convenience, we set $\sigma^1(x) = \sqrt{a}$ for $x \notin \overline{U}$. Therefore,

$$\Psi(X^p_{t\wedge\tau_U^{out}}) \geq M_{t\wedge\tau_U^{out}} - b(t \wedge \tau_U^{out}) \tag{13}$$

with $b = \sup_{x\in\overline{U}} |L\Psi(x)|$, and

$$\langle M\rangle_t = \sum_{j=1}^{m} \int_0^t \sigma^j(X_s^p)^2 ds \geq at.$$

By Dubins-Schwarz Theorem (see e.g [14], Theorem 5.13) there exists a Brownian motion (β) such that $M_t = \beta_{\langle M\rangle_t}$ for all $t \geq 0$. Thus, for all $\varepsilon > 0$

$$\sup_{0\leq t\leq\varepsilon} (M_t - bt) \geq \sup_{0\leq t\leq\varepsilon} (\beta_{\langle M\rangle_t} - \frac{b}{a}\langle M\rangle_t) \geq \sup_{0\leq t\leq a\varepsilon} (\beta_t - \frac{b}{a}t).$$

Let A_ε be the event that the right hand side of the last inequality is positive. We claim that A_ε has probability one. Indeed, by Blumenthal's zero-one law, the event $\cap_{n\geq 1} A_{1/n}$ has either probability one or zero. But

$$\mathbb{P}(A_\varepsilon) \geq \mathbb{P}(\beta_{a\varepsilon} > b\varepsilon) = \mathbb{P}(\beta_1 > \frac{b}{\sqrt{a}}\sqrt{\varepsilon}),$$

showing that $\liminf_{n\to\infty} \mathbb{P}(A_{1/n}) \geq 1/2$. Thus $\mathbb{P}(\cap_{n\geq 1} A_{1/n})$, hence $\mathbb{P}(A_{1/n})$, equals 1.

Now, using (13), it follows that $\tau_K^{out} \leq \varepsilon$ almost surely on the event $\{\tau_U^{out} > \varepsilon\}$. Thus $\mathbb{P}_p(\tau_K^{out} = 0) = 1$ because $\mathbb{P}_p(\tau_U^{out} > 0) = 1$.

The second statement of $(iii), (a)$ follows from the strong Markov property, valid since $(P_t)_{t\geq 0}$ is Feller (see e.g [14], Theorem 6.17), as follows. For all $x \in D$,

$$\mathbb{P}_x(\tau_D^{out} = \tau_K^{out}) = \mathbb{E}_x(\mathbb{P}_{X_{\tau_D^{out}}}(\tau_K^{out} = 0)) = 1.$$

b) Let $x \in D$. The property $\mathbb{P}_x(\tau_D^{out} = \tau_K^{out}) = 1$ implies that $P_t^D f(x) = P_t^K f(x)$ for all $f \in C_b(D)$. Hence, by (i) and (ii),

$$\lim_{y\to x} P_t^D f(y) = P_t^D f(x)$$

for all $f \in C_0(D), f \geq 0$. If now $f \in C_0(D)$ it suffices to write $f = f^+ - f^-$ with $f^+ = \max(f, 0)$ and $f^- = (-f)^+$.

c) Write $\tau_D^{x,out}$ for $\inf\{t \geq 0 : X_t^x \notin D\}$. Again, the property $\mathbb{P}_x(\tau_D^{out} = \tau_K^{out}) = 1$ combined with Lemma 17, imply that, almost surely, the maps $x \in D \to \tau_D^{x,out}$, and $x \in D \to \int_0^{\tau_D^{x,out}} f(X_s^x)ds$ are continuous for all $f \in C_b(D)$. Also,

$$\sup_{x\in D} \mathbb{E}[\int_0^{\tau_D^{x,out}} f(X_s^x)ds]^2] \leq \|f\|^2 \sup_{x\in D} \mathbb{E}_x[(\tau_D^{out})^2] < \infty$$

where the last inequality follows from Proposition 4. This shows that the family $(\int_0^{\tau_D^{x,out}} f(X_s^x)ds)_{x\in D}$ is uniformly integrable. Thus, $x \in D \to \mathbb{E}(\int_0^{\tau_D^{x,out}} f(X_s^x)ds) = G^D f(x)$ is continuous.

To conclude, observe that $|G^D f(x)| \leq \|f\| G^D \mathbf{1}_D(x)$ and that $G^D \mathbf{1}_D(x) = \mathbb{E}(\tau_x^{out,D})$. As $x \to p \in \partial D$ this last quantity converges to $\mathbb{E}(\tau_p^{out,D}) = 0$.
□

3.3 Proof of Theorem 2

Consequences of Hörmander conditions

We assume throughout all this subsection that the conditions of Theorem 1 hold and that the weak Hörmander condition is satisfied at every $x \in K$.

Lemma 19 *The operator G^D is a compact operator on $C_0(D)$.*

Proof: Let L be the differential operator defined, for g smooth, by

$$Lg = S^0(g) + \frac{1}{2}\sum_{j=1}^{m}(S^j)^2(g).$$

Consider the Dirichlet problem

$$\begin{cases} Lg = -f \text{ on } D \text{ (in the sense of distributions)} \\ g|_{\partial D} = 0. \end{cases} \tag{14}$$

We claim that for every $f \in C_b(K)$ there exists a solution to this problem given as

$$g(x) = G^D f(x)$$

for all $x \in D$.

Assume the claim is proved. By Theorem 18 in [21], there exists $0 < \alpha < 1$ depending only on the family $S^0, S^1, \ldots, S^m$ such that if $f \in L^\infty$ and $Lg = f$ (in the sense of distributions), then g is α-Hölder. Thus, $G^D(C_0(D)) \subset C_0^\alpha(D)$, the set of $f \in C_0(D)$ that are α-Hölder. Because G^D is a bounded operator on $C_0(D)$, it has a closed graph in $C_0(D) \times C_0^\alpha(D)$. Hence, by the closed graph theorem, it is a bounded operator from $C_0(D)$ into $C_0^\alpha(D)$. Compactness then follows from Ascoli's theorem.

We now pass to the proof of the claim.

By a theorem of Bony ([4], Théorème 5.2), for any $a > 0$ and $f \in C_b(K)$, the Dirichlet problem:

$$\begin{cases} Lg - ag = -f \text{ on } D \text{ (in the sense of distributions)} \\ g|_{\partial D} = 0; \end{cases} \tag{15}$$

has a unique solution, call it g_a, continuous on K. Furthermore, if f is smooth on D so is g_a. Note that the assumptions required for this theorem are implied by condition (ii) of Theorem 1 and the weak Hörmander condition.

Suppose that f is smooth on D (so that g_a is smooth on D). Then by Ito's formulae

$$(e^{-at\wedge\tau_D^{out}} g_a(X_{t\wedge\tau_D^{out}}) + \int_0^{t\wedge\tau_D^{out}} e^{-as} f(X_s)ds)_{t\geq 0}$$

is a local martingale. Being bounded, it is a uniformly integrable martingale. Thus, taking the expectation and letting $t \to \infty$, we get that

$$\int_0^\infty e^{-as} P_s^D f(x)ds = \mathsf{E}_x(\int_0^{\tau_D^{out}} e^{-as} f(X_s)ds) = g_a(x). \tag{16}$$

In particular, $G^D f(x) = \lim_{a\to 0} g_a(x)$, where the convergence is uniform by Proposition 4.

For every smooth test function Φ with compact support in D

$$\langle g_a, L^*\Phi\rangle - a\langle g_a, \Phi\rangle = -\langle f, \Phi\rangle$$

where $\langle h, \Phi\rangle = \int h(x)\Phi(x)dx$. Letting $a \to 0$, we get that $\langle G^D f, L^*\Phi\rangle = -\langle f, \Phi\rangle$, that is

$$LG^D(f) = -f,$$

in the sense of distributions. This proves that, for f smooth on D, the solution to (14) is $G^D(f)$. If now f is only continuous, let (f_n) be smooth (on a neighborhood of K) with $f_n \to f$ uniformly on K. Then $G^D(f_n) \to G^D(f)$ uniformly; and, by the same argument as above, $G^D(f)$ solves (14) in the sense of distributions. □

The proof of the next two lemmas are similar to the proof of Corollary 5.4 in [2]. For convenience we provide details.

Lemma 20 *Let $p \in D$ be such that $S^i(p) \neq 0$ for some $i \in \{1, \ldots, m\}$. Then, there exist disjoint open sets $U, V \subset D$ with $p \in U$ and a non-trivial measure ξ on V (i.e $\xi(V) > 0, \xi(M \setminus V) = 0$) such that for all $x \in U$,*

$$G^D(x, \cdot) \geq \xi(\cdot).$$

Proof: We first assume that for all $x \in D$ there is some $i \in \{1, \ldots, m\}$ such that $S^i(x) \neq 0$. By Theorem 6.1 in [4], this condition (combined with condition (ii) of Theorem 1 and the weak Hörmander assumption), imply that, for all $a > 0$, there exists a map $K_a : \overline{D}^2 \mapsto \mathbb{R}_+$ smooth on $D^2 \setminus \{(x, x) : \ x \in D\}$ such that for all $f \in C_b(K)$, and $x \in D$

$$\int_0^\infty e^{-as} P_s^D f(x) ds = \int_D K_a(x, y) f(y) dy.$$

To be more precise, Theorem 6.1 in [4] asserts that the solution to the Dirichlet problem (15) can be written under the form given by the right hand side of this equality. On the other hand, we have shown in the proof of Lemma 19, that this solution is given by the left hand side.

Given $p \in D$, choose $q \neq p$ such that $K_a(p, q) > 0$. Such a q exists, for otherwise, we would have $\int_0^\infty e^{-as} P_s^D \mathbf{1}_D(p) ds = 0$. That is $\tau_D^{out} = 0, \mathbb{P}_p$ almost surely. By continuity of K_a off the diagonal, there exist disjoint neighborhoods U, V of p and q and some $c > 0$ such that $K_a(x, y) \geq c$ for all $x \in U, y \in V$. Thus, for all $x \in U$,

$$G^D(x, \cdot) \geq \int_0^\infty e^{-as} P_s^D(x, \cdot) \geq cLeb(V \cap \cdot)$$

where Leb stands for the Lebesgue measure on $\mathbb{R}^n$.

We now pass to the proof of the Lemma. Using a local chart around p we can assume without loss of generality that $p = 0, S^1(0) \neq 0$ and $\frac{S^1(0)}{\|S^1(0)\|} = e_1$, where $(e_1, \ldots, e_n)$ stands for the canonical basis on $\mathbb{R}^n$. Let $D_\varepsilon = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i| < \varepsilon\}$. For $\varepsilon > 0$ small enough $S^1(x) \neq 0$ for all $x \in D_\varepsilon$ and for all $x \in \overline{D_\varepsilon}$ there is a vector u normal to $\overline{D_\varepsilon} \setminus D_\varepsilon$ (in the sense of [4]) such that $\langle S^1(x), u \rangle \neq 0$. The preceding reasoning can then be applied with D_ε in place of D. Thus, for some disjoint open sets $U, V \subset D_\varepsilon$ with $p \in U$ and for all $x \in U$,

$$G^D(x, \cdot) \geq G^{D_\varepsilon}(x, \cdot) \geq cLeb(V \cap \cdot).$$

□

Lemma 21 *Let $p \in D$ be such that:*

(i) *The Hörmander condition holds at p;*

(ii) *There exists a neighborhood $V \subset D$ of p such that p is D-accessible from all $x \in V$.*

Then, there exist an open set $U \subset V$, a non trivial measure ξ on U and positive numbers $T > \varepsilon > 0$ such that for all $x \in U, T - \varepsilon \le t \le T$ $P_t^D(x, \cdot) \ge \xi(\cdot)$.

Proof: Replacing V by a smaller neighborhood if necessary we can assume that the Hörmander condition holds at every $x \in V$. By ([12], Theorem 3 and its proof), there exists a nonnegative map $p_t^V(x, y)$ smooth in the variables $(t, x, y) \in \mathbb{R}_+^* \times V \times V$ such that

$$P_t^V(x, dy) = p_t^V(x, y)dy,$$

where $P_t^V(x, \cdot)$ stands for the law of X_t^x killed at $D \setminus V$. Choose $q \in V$ and $T > 0$ such that $p_T^V(p, q) := c > 0$. Then, by continuity, there exist $\varepsilon > 0$ and neighborhoods V_1, V_2 of p and q such that

$$p_t^V(x, y) \ge c > 0$$

for all $(t, x, y) \in [T - \varepsilon, T + \varepsilon] \times V_1 \times V_2$. By D-accessibility of p from q, there exist $s \ge 0, \delta > 0$ such that $P_s^D(q, V_1) > \delta$. Now, by the continuity property $(iii), (c)$ of Lemma 18, the set

$$U = \{y \in V_2 : \ P_s^D(y, V_1) > \delta\}$$

is an open neighborhood of q. The proof of this fact is verbatim the same as the proof written for G^D in the proof of Step 1 of Theorem 12. Thus, for all $x \in U, T - \varepsilon \le t \le T + \varepsilon$, and $A \subset U$ a Borel set,

$$P_{t+s}^D(x, A) \ge \int_{V_1} P_s^D(x, dy) P_t^D(y, A) \ge \int_{V_1} P_s^D(x, dy) \int_A p_t^V(y, z) dz$$

$$\ge \delta c Leb(A).$$

□

Lemma 22 *Let μ be a QSD. Then μ has a smooth density with respect to the Lebesgue measure on D. If furthermore the strong Hörmander conditions holds at every $x \in K$, this density is positive.*

Proof: Let Φ be a smooth function with compact support in D. By Ito's formulae $(\Phi(X_{t\wedge\tau_D^{out}}) - \int_0^{t\wedge\tau_D^{out}} L\Phi(X_s)ds)_{t\geq 0}$ is a local martingale, whose quadratic variation is in L^2, hence a true martingale. Thus, taking the expectation and letting $t\to\infty$, it comes that $G^D(L\Phi)(x) = -\Phi(x)$ for all $x\in D$. Let μ be a QSD with rate λ. Then $-\mu(\Phi) = \mu G^D(L\Phi) = \frac{1}{\lambda}\mu(L\Phi)$. This shows that $L^*\mu + \lambda\mu = 0$ on D in the sense of distributions. Now,

$$L^* f = \tilde{S}^0 f + \frac{1}{2}\sum_{j=1}^{m}(S^j)^2 f + Tf$$

where T is a smooth function and $S^0 + \tilde{S}^0 \in Span(S^1,\ldots,S^m)$. Therefore L^* satisfies the weak Hörmander property. By Hörmander Theorem [11], it is hypoelliptic. This implies that μ has a smooth density.

For $a>0$, set $L_a^* f = L^* f - af$ and choose a sufficiently large so that $L_a^* 1 = T - a < 0$. If the strong Hörmander is satisfied at every point $x\in K$, the same is true for L_a^*. Now, $L_a^*(-\mu) = (\lambda + a)\mu \geq 0$. Therefore, by application of Bony's maximum principle ([4], Corollary 3.1), if the density of $-\mu$ vanishes at some $x\in D$, it has to be zero on D. This is impossible because μ is a probability measure. □

Proof of Theorem 2

We now assume that conditions $(i),(ii),(iii)$ of Theorem 2 hold. By Lemma 19 and Corollary 9, there exists a positive right eigenfunction for G^D. By Lemma 20 and Lemma 11, (P_t^D) is irreducible. Thus, according to Theorem 10, it has a unique QSD. Such a QSD has a smooth density by Lemma 22. This prove the first part of Theorem 2.

The last part follows from Lemma 21 and Theorem 12.

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