Quasi-compactness criterion for strong Feller kernels with an application to quasi-stationary distributions

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Let (E,d) be a separable metric space, which is locally compact (i.e. such that $E = \cup_{n\geq 0} L_n$ for some non-decreasing sequence of compact subsets $(L_n)_{n\geq 0}$). Given $\psi : E \to (0,+\infty)$, we define the complex vector space

$$\mathscr{B}_b(\psi) = \{f : E \to \mathbb{C}, f \text{ measurable and } \|f/\psi\|_\infty < +\infty\},$$

endowed with the complete norm

$$\|f\|_\psi = \|f/\psi\|_\infty.$$

Following Definition II.1 in [9], a bounded operator Q on a Banach space B is said to be quasi-compact if B can be decomposed into two Q-invariant closed subspaces

$$B = F \oplus H,$$

where $r(Q_{|H}) < r(Q)$ (with $r(Q_{|H})$ the spectral radius of Q restricted to H and $r(Q)$ the spectral radius of Q), while $\dim F < +\infty$ and each eigenvalue of $Q_{|F}$ has modulus $r(Q)$. The essential spectral radius of Q (see for instance Definition XIV.1 in [9]) is the greatest lower bound of $r(Q)$ and of the real numbers $\rho \geq 0$ for which there exists a decomposition into closed Q-invariant subspaces

$$B = F_\rho \oplus H_\rho$$

where $\dim F_\rho < +\infty$ and $Q_{|F_\rho}$ has only eigenvalues of modulus $\geq \rho$, while $r(Q_{|H_\rho}) < \rho$. In particular, the quasi-compactness of Q equivalent to *"the essential spectral radius of Q is strictly smaller than the spectral radius of Q"*.

The aim of this note is to provide a sufficient criterion for the quasi-compactness on $\mathscr{B}_b(\psi)$ of operators defined by a strong Feller kernel on E, with an explicit upper bound on the essential spectrum. Our main tool is Ionescu-Tulcea's theorem, as stated in Theorem XIV.3 in Hennion and Hervé's book [9]. Our criterion is stated and proved in Section 1, where we first consider quasi-compactness criteria for strong Feller sub-Markov kernels on $\mathscr{B}_b(E) := \mathscr{B}_b(\mathbb{1}_E)$ (Section 1.1) and then of non-conservative kernels

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acting on a space $\mathscr{B}_b(\psi)$ (Section 1.2). In Section 2, we apply this criterion to the study of quasi-stationary distributions. More precisely, we show that a strong Feller assumption, a local irreducibility assumption and a Lyapunov type assumption entail a weak form of Harnack inequality. Since this inequality is a key element of the abstract results presented in [3] and is usually difficult to check, our result greatly simplifies the study of quasi-stationary distributions for strong Feller Markov processes.

Note that the theory of quasi-compactness of kernel operators has already been applied to quasi-stationary theory (see Section 4 in [10], and in particular the key Perron-Frobenius result Theorem A therein, which we won't be in a position to use). We also refer the reader to the recent papers [5, 6, 2]. Quasi-compactness results for operator on $\mathscr{B}_b(\psi)$ spaces also has applications in Pólya urn's theory (see for instance [11]) and fluctuations results for Markov chains (see for instance [9] and references therein). We also refer the reader to [8] for the study and formulas of the essential spectral radius, with a link to the classical Doeblin criterion for the ergodicity of Markov chains.

1 Quasi-compactness of strong Feller kernel operators

1.1 Quasi-compactness on $\mathscr{B}_b(E)$

Let P be a sub-Markov kernel on E (i.e. $P\mathbb{1}_E \leq \mathbb{1}_E$) which enjoys the strong Feller property (for all $f \in \mathscr{B}_b(E)$, Pf is continuous on E). We assume that

$$r_\infty := \lim_{n_0\to+\infty} \lim_{n\to+\infty} \sup_{x\in E\setminus L_n} P\mathbb{1}_{E\setminus L_{n_0}}(x) < r(P), \tag{1}$$

where $r(P)$ is the spectral radius of P on $\mathscr{B}_b(E)$ (endowed with the supremum norm $\|\cdot\|_\infty$).

We show the following

Theorem 1. *The kernel operator P is quasi-compact on $\mathscr{B}_b(E)$, with essential spectral radius bounded above by r_∞.*

Remark 1. Let π be a probability measure on E such that $\delta_x P \ll \pi$ for all $x \in E$ (the existence of π is guaranteed by Lemma 5.23 in [1]). One can adapt the arguments of this section to obtain the quasi-compactness of P on $L^\infty(\pi)$, the set of π-almost surely bounded functions on E.

Proof of Theorem 1. We will make use of Theorem XIV.3 in [9], with $Q = P^2$, $(\mathscr{B}, \|\cdot\|) = (\mathscr{B}_b(E), \|\cdot\|_\infty)$. The semi-norm $|||\cdot|||$ in our case is a norm on the set $C(E)$ (of bounded continuous functions) for the uniform convergence on compact sets. In order to define it, we first need a well chosen collection of compact sets, whose existence is guaranteed by

Lemma 2. *There exists a sequence of compact subsets $(K_n)_{n\geq 0}$ such that, for all $n \geq 0$, $K_n \supset L_n$ and*

$$\sup_{x\in K_n} P\mathbb{1}_{E\setminus K_{n+1}}(x) \leq 1/2^{n+1}. \tag{2}$$

Proof of Lemma 2. Let K be any compact subset of E. Then $f_n = P\mathbb{1}_{E\setminus(K\cup L_n)}$ defines a sequence of continuous functions (since P is strongly Feller) which converges monotonically and pointwisely to 0. In particular, according to Dini's theorem, $(f_n)_n$ converges uniformly to 0 on K and hence, for any $\varepsilon > 0$, there exists $n \geq 0$ such that

$$\sup_{x\in K} P\mathbb{1}_{E\setminus(K\cup L_n)}(x) \leq \varepsilon.$$

This allows to construct K_n recursively, starting with $K_0 = L_0$. □

Take a sequence $(K_n)_n$ as in Lemma 2 and define

$$|||f||| = \sum_{n\geq 0} \frac{\|f\mathbb{1}_{K_n}\|_\infty}{2^n}.$$

In order to apply Theorem XIV.3 in [9], we need to check that

i. $Q(\{f : f \in \mathscr{B}_b(E),\ \|f\|_\infty \leq 1\})$ is relatively compact in $(\mathscr{B}_b(E), |||\cdot|||)$;

ii. $\exists M > 0$ such that $|||Qf||| \leq M|||f|||$ for all $f \in \mathscr{B}_b(E)$;

iii. $\exists r < r(Q) = r(P^2)$ (this is a consequence of Gelfand's formula) and $R \geq 0$ such that $\|Qf\|_\infty \leq r\|f\|_\infty + R|||f|||$ for all $f \in \mathscr{B}_b(E)$.

Once this is proved, Theorem XIV.3 entails that Q is quasi-compact on $\mathscr{B}_b(E)$, with essential spectral radius smaller than r, so that, by the spectral mapping theorem, P is quasi-compact on $\mathscr{B}_b(E)$, with essential spectral radius smaller than $\sqrt{r}$. Since we will show that r can be chosen arbitrarily close to r_∞^2, this entails Theorem 1.

It remains to prove i, ii and iii.

Proof of i. This is a direct consequence of the fact that Q is actually ultra-Feller (see Proposition 5.22 p.89 in [1], see also Corollary 2.4 in [14]).

Proof of ii. For all $n \geq 0$ and all $x \in K_n$, we have

$$\begin{aligned}
Pf(x) &= P(f\mathbb{1}_{K_{n+1}})(x) + \sum_{k\geq n+1} P(f\mathbb{1}_{K_{k+1}\setminus K_k})(x) \\
&\leq \|f\mathbb{1}_{K_{n+1}}\|_\infty + \sum_{k\geq n+1} P(\mathbb{1}_{E\setminus K_k})(x)\|f\mathbb{1}_{K_{k+1}}\|_\infty \\
&\leq \|f\mathbb{1}_{K_{n+1}}\|_\infty + \sum_{k\geq n+1} \frac{\|f\mathbb{1}_{K_{k+1}}\|_\infty}{2^{k+1}} \\
&\leq \|f\mathbb{1}_{K_{n+1}}\|_\infty + |||f|||.
\end{aligned}$$

In particular, we deduce that

$$\frac{\|Pf\mathbb{1}_{K_n}\|_\infty}{2^n} \leq 2\frac{\|f\mathbb{1}_{K_{n+1}}\|_\infty}{2^{n+1}} + \frac{|||f|||}{2^n}.$$

Summing over $n \geq 0$ concludes, we deduce that

$$|||Pf||| \leq 4|||f|||.$$

Applying this result to Pf instead of f, we deduce that ii. holds true with $M = 16$.

Proof of iii. Fix $\varepsilon > 0$ such that $r_\varepsilon = r_\infty + \varepsilon < r(P)$ and let $n_0, n_1 \geq 0$ such that

$$\sup_{x \in E \setminus L_{n_1}} P\mathbb{1}_{E\setminus L_{n_0}} < r_\varepsilon.$$

Then, for all $x \notin K_{n_1}$, we have $x \notin L_{n_1}$, and hence, for all $f \in \mathscr{B}_b(E)$,

$$\begin{aligned}|Pf(x)| &\leq |P(f\mathbb{1}_{L_{n_0}})(x)| + |P(f\mathbb{1}_{E\setminus L_{n_0}})(x)| \\ &\leq \|f\mathbb{1}_{L_{n_0}}\|_\infty + r_\varepsilon \|f\|_\infty \leq 2^{n_0} \left|\left|\left| f \right|\right|\right| + r_\varepsilon \|f\|_\infty.\end{aligned}$$

Also, for all $x \in K_{n_1}$, we have (see proof of ii.)

$$|Pf(x)| \leq \|f\mathbb{1}_{K_{n_0+1}}\|_\infty + \left|\left|\left| f \right|\right|\right| \leq (2^{n_0+1}+1)\left|\left|\left| f \right|\right|\right|.$$

We deduce that

$$\|Pf\|_\infty \leq (2^{n_0+1}+1)\left|\left|\left| f \right|\right|\right| + r_\varepsilon \|f\|_\infty$$

and hence that

$$\|Qf\|_\infty \leq (2^{n_0+1}+1)\left|\left|\left| Pf \right|\right|\right| + r_\varepsilon\|Pf\|_\infty \leq (2^{n_0+1}+1)M\left|\left|\left| f \right|\right|\right| + r_\varepsilon(2^{n_0+1}+1)\left|\left|\left| f \right|\right|\right| + r_\varepsilon^2\|f\|_\infty.$$

Since $r_\varepsilon^2 < r(P)^2 = r(Q)$, this concludes the proof of iii. with $r = r_\varepsilon^2$.

□

1.2 Quasi-compactness on $\mathscr{B}_b(\psi)$

Let P be a kernel on E such that, for some continuous function $\psi : E \to (0, +\infty)$, $\|P\psi\|_\psi < +\infty$, so that P acts as a bounded linear operator on the Banach space $\mathscr{B}_b(\psi)$ with the strong ψ-Feller property:

$$\text{(strong } \psi\text{-Feller)} \ \forall f \in \mathscr{B}_b(\psi),\ Pf \text{ is continuous on } E.$$

Actually the strong ψ-Feller property can be deduced, in some situations, from the strong Feller property, as explained in Proposition 4 below.

Remark 2. As will appear in the proof, the assumption that P is ψ-Feller with ψ continuous could actually be replaced with the assumption that P^ψ is strong Feller.

We also assume that

$$r_{\psi,\infty} = \lim_{n_0\to+\infty} \lim_{n\to+\infty} \sup_{x\in E\setminus L_n} \frac{P(\psi\mathbb{1}_{E\setminus L_{n_0}})(x)}{\psi(x)} < r_\psi(P), \tag{3}$$

where $r_\psi(P)$ is the spectral radius of P on $\mathscr{B}_b(\psi)$. Proposition 5 below provides a simple criterion to obtain lower bounds on $r_\psi(P)$.

Theorem 3. *Under the assumptions of this section, the semi-group P is quasi-compact on $\mathscr{B}_b(\psi)$, with essential spectral radius bounded above by $r_{\psi,\infty}$.*

Proof of Theorem 3. We define the sub-Markov semi-group P^ψ on E by

$$P^\psi f(x) = \frac{1}{\|P\psi\|_\psi \psi(x)} P(f\psi),\ \forall f \in \mathscr{B}_b(E).$$

The fact that ψ is continuous and P is strongly ψ-Feller entails that P^ψ is strongly Feller.

In addition, the spectral radius of P^ψ on $\mathscr{B}_b(E)$ is

$$r(P^\psi) = \frac{r_\psi(P)}{\|P\psi\|_\psi}$$

and hence (3) implies that

$$r_\infty := \lim_{n_0\to+\infty} \lim_{n\to+\infty} \sup_{x\in E\setminus L_n} P^\psi(\mathbb{1}_{E\setminus L_{n_0}})(x) = \frac{r_{\psi,\infty}}{\|P\psi\|_\psi} < \frac{r_\psi(P)}{\|P\psi\|_\psi} = r(P^\psi)$$

Applying Theorem 1 to P^ψ, we deduce that the sub-Markov kernel P^ψ is quasi-compact on $\mathscr{B}_b(E)$, with essential spectral radius bounded above by $r_\infty = \frac{r_{\psi,\infty}}{\|P\psi\|_\psi}$. We deduce that P is quasi-compact on $\mathscr{B}_b(\psi)$ with essential spectral radius bounded above by $r_{\psi,\infty}$. This concludes the proof of Theorem 3. □

In the following proposition, we provide a criterion ensuring that a strong Feller kernel P satisfies the strong ψ-Feller property. We say that $\psi = o(V)$ if there exists a non-increasing function $h : [0,+\infty) \to (0,+\infty)$ which vanishes at infinity and such that $\psi/V \le h\circ V$. We say that $V : E \to (0,+\infty)$ is locally bounded if, for all $x \in E$, there exists $\eta > 0$ such that

$$\sup_{y\in E,\, d(x,y)<\eta} V(y) < +\infty.$$

Proposition 4. *Assume that P is a strong Feller kernel on E and that, for some locally bounded function $V : E \to (0,+\infty)$, $\|PV\|_V < +\infty$. Then, for any function $\psi : E \to (0,+\infty)$ such that $\psi = o(V)$ and such that $\|P\psi\|_\psi < +\infty$, P satisfies the strong ψ-Feller property.*

Proof. Let $f \in \mathscr{B}_b(\psi)$, and fix $\varepsilon > 0$ and $x \in E$. Then, for all $M \ge 1$ and all $y \in E$,

$$\begin{aligned}\left|Pf(x) - Pf(y)\right| \le |P(f\mathbb{1}_{V\le M})(x) - P(f\mathbb{1}_{V\le M})(x)|\\ + |P(f\mathbb{1}_{V>M})(x)| + |P(f\mathbb{1}_{V>M})(y)| \quad (4)\end{aligned}$$

Since V is locally bounded, there exists $\eta_0 > 0$ and $C_0 > 0$ such that, for all $y \in E$ such that $d(x,y) < \eta_0$, $V(y) \le C_0$. Then, for all such y, we have

$$\begin{aligned}|P(f\mathbb{1}_{V>M})(y)| &\le \|f\|_\psi P(\psi\mathbb{1}_{V>M})(y)\\ &\le \|f\|_\psi P(h\circ V\mathbb{1}_{V>M} V)(y)\\ &\le \|f\|_\psi h(M)\, PV(y)\\ &\le \|f\|_\psi h(M)\, \|PV\|_V\, V(y)\\ &\le \|f\|_\psi h(M)\, \|PV\|_V\, C_0.\end{aligned}$$

Since $h(M) \to 0$ when $M \to +\infty$, we deduce that there exists M large enough such that, for all $y \in E$ such that $d(x,y) < \eta_0$,

$$|P(f \mathbb{1}_{V>M})(x)| + |P(f \mathbb{1}_{V>M})(y)| \leq \varepsilon/2. \tag{5}$$

In addition, since $|f \mathbb{1}_{V\leq M}| \leq \|f\|_\psi \, h(0) M$, we have $f \mathbb{1}_{V\leq M} \in \mathscr{B}_b(E)$. Since P is strong Feller, we deduce that $P(f \mathbb{1}_{V\leq M})$ is continuous. Hence there exists $\eta_1 > 0$ such that, for all $y \in E$ such that $d(x,y) < \eta_1$,

$$|P(f \mathbb{1}_{V\leq M})(x) - P(f \mathbb{1}_{V\leq M})(x)| \leq \varepsilon/2. \tag{6}$$

Taking $\eta = \eta_0 \wedge \eta_1$, we deduce from (4), (5) and (6) that, for all $y \in E$ such that $d(x,y) < \eta$, we have

$$\left| Pf(x) - Pf(y) \right| \leq \varepsilon.$$

This entails that Pf is continuous and hence that P satisfies the strong ψ-Feller property. □

The following proposition provides a simple way to obtain a lower bound for $r_\psi(P)$. The condition is very similar to the second part of condition of (E2) in [3], and we refer the reader to the methods provided therein to build the function φ.

Proposition 5. *Let $\theta \geq 0$. If there exists a non-negative non-zero function $\varphi \in \mathscr{B}_b(\psi)$ such that $P\varphi \geq \theta\varphi$, then $r_\psi(P) \geq \theta$.*

Proof. We assume without loss of generality that $\|\varphi\|_\psi \leq 1$. Let $x \in E$ such that $\varphi(x) > 0$. Then we have, for all $n \geq 1$,

$$P^n \varphi(x) \geq \theta^n \varphi(x) \geq \theta^n \psi(x) \frac{\varphi(x)}{\psi(x)}.$$

In particular,

$$\left(\left\| P^n \varphi \right\|_\psi \right)^{1/n} \geq \theta \left(\frac{\varphi(x)}{\psi(x)} \right)^{1/n}.$$

By Gelfand's formula, the lim inf as $n \to +\infty$ of the left hand side is smaller than $r_\psi(P)$, while the right hand side converges to θ. We deduce that $r_\psi(P) \geq \theta$, which concludes the proof. □

2 Application to quasi-stationary distributions

In this section, we consider a discrete time Markov process $(X_n)_{n\in\mathbb{N}}$ on $E \cup \{\partial\}$, where $(E \cup \{\partial\}, d)$ is a locally compact separable metric space and ∂ is an isolated point outside of E. We denote the law of X with initial distributions μ on E by $\mathbb{P}_\mu$ and its associated expectation by $\mathbb{E}_\mu$. As usual, we write $\mathbb{P}_x$ and $\mathbb{E}_x$ when μ is the Dirac measure δ_x at $x \in E$. We assume that ∂ is an absorbing point, meaning that, for all $x \in E \cup \partial$, $X_n = \partial$ for all $n \geq \tau_\partial := \inf\{k \geq 0,\ X_k = \partial\}$ $\mathbb{P}_x$ almost surely.

We consider the following assumption, where *"K is irreducible"* means that, for all $x, y \in K$ and all neighborhood V of y, there exists $t > 0$ such that $\mathbb{P}_x(X_t \in V) > 0$.

Assumption (A). There exist a positive integer n_1, positive real constants $\theta_1, \theta_2, c_1, c_2$, a continuous function $\varphi_1 : E \to [1, +\infty)$, a non-negative non-zero function $\varphi_2 \in \mathscr{B}_b(E)$ and a probability measure ν on a subset $K \subset E$ such that

(A1) *(Local Dobrushin coefficient).* $\forall x \in K$,

$$\mathbb{P}_x(X_{n_1} \in \cdot) \geq c_1 \nu(\cdot \cap K).$$

(A2) *(Global Lyapunov criterion).* We have $\theta_1 < \theta_2$, $\inf_K \varphi_2 > 0$ and, for all $x \in E$,

$$\mathbb{E}_x(\varphi_1(X_1) \mathbb{1}_{1<\tau_\partial}) \leq \theta_1 \varphi_1(x) + c_2 \mathbb{1}_K(x),$$
$$\mathbb{E}_x(\varphi_2(X_1) \mathbb{1}_{1<\tau_\partial}) \geq \theta_2 \varphi_2(x).$$

(A3) *(compactness, irreducibility and strong Feller property).* We assume that K is compact and irreducible, and that the kernel P from E to E, defined by

$$Pf(x) = \mathbb{E}_x(f(X_1) \mathbb{1}_{1<\tau_\partial}), \ \forall x \in E \text{ and } f \in \mathscr{B}_b(\varphi_1),$$

satisfies the strong φ_1-Feller property.

(A4) *(Aperiodicity).* For all $x \in K$, there exists $n_4(x)$ such that, for all $n \geq n_4(x)$,

$$\mathbb{P}_x(X_n \in K) > 0.$$

Before stating the main result of this section, we emphasize that Assumption (A) is very close to Assumption (E) in [3], with the main difference being that (A3) is replaced therein by (E3), which corresponds to the following Harnack type inequality :

$$\inf_{n \geq 0} \frac{\inf_{x \in K} \mathbb{P}_x(n < \tau_\partial)}{\sup_{x \in K} \mathbb{P}_x(n < \tau_\partial)} > 0. \tag{7}$$

The following result shows that Assumption (A) in fact entails that Condition (E3) is satisfied, and hence that Assumption (A) implies that Assumption (E) holds true. This is crucial since, in many situations, checking condition (E3) can be quite difficult, while checking the strong Feller property is straightforward.

Proposition 6. *Assume that Conditions (A1), (A2) and (A3) hold true. Then* (7) *holds true.*

In addition, Assumption (A) entails spectral interpretations of the parameters θ_1 and θ_2 in Assumption (E) in the context of strong Feller kernels. The following result yields a upper bound on the essential spectral radius of P and a lower bound on the spectral radius of P on $\mathscr{B}_b(\varphi_1)$. The proof is identical to the one of the same property for R (as defined in the proof of Proposition 6 below), so we skip it.

Proposition 7. *If Conditions (A2) and (A3) hold true, then the essential spectral radius of P on $\mathscr{B}_b(\varphi_1)$ is smaller than or equal to θ_1, and the spectral radius of P on $\mathscr{B}_b(\varphi_1)$ is greater than or equal to θ_2.*

In the following statement, we make use of the set $E' = \{x \in E,\ \exists k \geq 0,\ P^k\varphi_2(x) > 0\}$, which obviously contains K. It is an immediate corollary of Proposition 6 and of [3].

Theorem 8. *If Assumption (A) holds true, then Condition (E) from [3] is satisfied. In particular, X admits a unique quasi-stationary distribution ν_{QS} satisfying $\nu_{QS}(\varphi_1) < +\infty$ and $\nu_{QS}(E') > 0$. In addition, $\nu_{QS}(K) > 0$ and there exist a constant $C > 0$ and a constant $\alpha \in (0,1)$ such that*

$$\left|\mathbb{E}_\mu(f(X_n) \mid n < \tau_\partial) - \nu_{QS}(f)\right| \leq C\alpha^n \|f\|_{\varphi_1} \frac{\mu(\varphi_1)}{\mu(\varphi_2)}, \quad \forall n \geq 0,\ \forall f \in \mathscr{B}_b(\varphi_1), \tag{8}$$

for all probability measure μ on E such that $\mu(\varphi_1) < \infty$ and $\mu(\varphi_2) > 0$.

In [3], several other results are deduced from the Assumption (E). In particular, we recall the following

Corollary 9. *Assume that Condition (A) holds true. Then there exist constants $C > 0$ and $\alpha \in (0,1)$, and a non-negative function $\eta : E \to [0,+\infty)$ such that η is positive on E', $\eta \in \mathscr{B}_b(\varphi_1)$ such that, for all probability measure μ on E such that $\mu(\varphi_1) < +\infty$,*

$$\left|\theta_0^{-n}\mathbb{E}_\mu(f(X_n)\mathbb{1}_{n<\tau_\partial}) - \mu(\eta)\nu_{QS}(f)\right| \leq C\alpha^n \|f\|_{\varphi_1}\mu(\varphi_1), \quad \forall f \in \mathscr{B}_b(\varphi_1), \tag{9}$$

where $\theta_0 \geq \theta_2 > \theta_1$ is the spectral radius of P on $\mathscr{B}_b(\varphi_1)$.

Finally, we emphasize that it is straightforward to adapt Assumption (A) to non-sub Markov operators P, with $\varphi_2 \in \mathscr{B}_b(\varphi_1)$ (instead of $\varphi_2 \in \mathscr{B}_b(E)$) and to $\varphi_1 : E \to (0,+\infty)$ (instead of $\varphi_1 : E \to [1,+\infty)$), by proceeding to φ_1 transforms of the operator. Indeed, for such an operator P, the operator Q defined by

$$Qf(x) = \frac{1}{\|P\varphi_1\|_{\varphi_1}\varphi_1(x)} P(\varphi_1 f)(x)$$

for all $x \in E$ and $f \in \mathscr{B}_b(E)$ is then a sub-Markov operator which satisfies Assumption (A) with $\varphi_2' = \varphi_2/\varphi_1 \in \mathscr{B}_b(E)$ instead of φ_2 and $\varphi_1' \equiv 1$ instead of φ_1. The details can be found in [4] in the context of Assumption (E) from [4].

We proceed now to the proof of Proposition 6. In [10], the authors provide a general application of quasi-compactness properties to the study of quasi-stationary distributions. In our situation, the operator P on $\mathscr{B}_b(\varphi_1)$ is not (in general) strictly positive - following the terminology of [13] - and thus the Perron-Frobenius type Theorem 7 therein, cited as Theorem A in [10], does not apply. We use a different approach to conclude, based on the study of a well chosen h-transform of the process. We emphasize that we do not use the aperiodicity assumption (A4), and hence that our result also applies to periodic processes. Adapting Theorem 8 to periodic processes is straightforward (although cumbersome) and working out the details of the adaptation is thus left to the interested reader.

Proof of Proposition 6. Consider the space E_K of points $y \in E$ accessible from K, defined as

$$E_K = \{y \in E \text{ s.t., for all } x \in K \text{ and all neighborhood } V \text{ of } y,\ \exists n \geq 0 \text{ s.t. } \mathbb{P}_x(X_n \in V) > 0\}.$$

Let us first prove that the set E_K is irreducible, meaning that, for all $x \in E_K$ and all $A \subset E \setminus E_K$, $\mathbb{P}_x(X_1 \in A) = 0$. Let $x \in E_K$ and $A \subset E \setminus E_K$. For any $n \geq 0$, let $B_n = A \cap L_n$. Then, by definition of E_K, each point $y \in B_n$ admits a neighborhood V_y such that $\mathbb{P}_x(X_1 \in V_y) = 0$. Since B_n is relatively compact, we deduce that there exists a finite sequence $y_1, \dots, y_k$ such that $B_n \subset \cup_{i=1}^k V_i$ and hence that $\mathbb{P}_x(X_1 \in B_n) = 0$. Since $A = \cup_{n\geq 0} B_n$, we deduce that $\mathbb{P}_x(X_1 \in A) = 0$, which concludes the proof.

We can thus consider the process Y defined as X restricted to E_K and, setting $\varphi_1^K = \varphi_{1|E_K}$, its associated kernel R satisfies

$$\forall x \in E_K, \forall f \in \mathscr{B}_b(\varphi_1^K),\ Rf(x) = Pf(x) = \mathbb{E}_x(f(X_1)\mathbb{1}_{1<\tau_\partial}),$$

with f extended to 0 outside of E_K. Since $K \subset E_K$ and since E_K is absorbing for the Markov chain X, we deduce that Assumption (A) also holds true for the process Y with E_K instead of E, φ_1^K instead of φ_1, and $\varphi_2^K := \varphi_{2|E_K}$ instead of φ_2.

In particular, R satisfies the strong φ_1^K-Feller property. In addition, according to Proposition 5 and the second line of (A2), we deduce that $r_{\varphi_1^K}(R) \geq \theta_2 > \theta_1$. Hence, using the first line of (A2) and redefining if necessary the sequence $(L_n)_{n\geq 0}$ so that $L_0 = K$, we deduce that (3) holds true. We are thus in position to apply Theorem 3, which entails that R is quasi-compact in $\mathscr{B}_b(\varphi_1^K)$, with essential spectral radius smaller than or equal to θ_1.

Applying Theorem 3 in [13] (considering the cone of non-negative functions in $\mathscr{B}_b(\varphi_1^K)$), we deduce that there exists $\eta \in \mathscr{B}_b(\varphi_1^K)$ with $\eta \geq 0$, $\eta \neq 0$, and such that

$$R\eta = r_{\varphi_1^K}(R)\eta.$$

Since R satisfies the strong φ_1^K-Feller property, we deduce that $R\eta$ and hence η are continuous on E_K. In particular, there exists a point $y \in E_K$ and a neighborhood V of y such that $\eta > 0$ on V. By definition of E_K, we know that, for all $x \in K$, there exists $n \geq 0$ such that $\mathbb{P}_x(Y_n \in V) = \mathbb{P}_x(X_n \in V) > 0$, and hence $\eta(x) = 1/r_{\varphi_1^K}(R)^n R^n \eta(x) > 0$. This implies that η is positive on K.

Now define the set $E'_K := \{x \in E_K,\ \eta(x) > 0\}$ (note that $K \subset E'_K$) and the η-transform Q of R on E'_K: setting $\psi = \varphi_1/\eta \mathbb{1}_{E'_K}$, it is defined as

$$\forall x \in E'_K,\ \forall f \in \mathscr{B}_b(\psi), \quad Qf(x) = \frac{1}{r_{\varphi_1^K}(R)\,\eta(x)} R(f\eta)(x).$$

It is well known and easy to check that Q is the transition kernel of a conservative Markov process on E'_K. In addition, Assumption (A) for R and the positiveness and boundedness of η on K entail that Q satisfies the classical Foster Lyapunov assumptions (see for instance [7] or the classical reference [12]). In particular, there exist a probability measure ν_S on E'_K and constants $C > 0$ and $\bar\alpha \in (0,1)$ such that, for all $x \in E'_K$, $n \geq 0$ and all $f \in \mathscr{B}_b(\psi)$,

$$\left|Q^n f(x) - \nu_S(f)\right| \leq C\psi(x)\,\|f\|_\psi\,\bar\alpha^n.$$

This implies that, for all $f \in \mathscr{B}_b(\varphi_1^K \mathbb{1}_{E'_K})$ and all $x \in E'_K$,

$$\left|(r_{\varphi_1^K}(R))^{-n} R^n(f)(x) - \nu_S(\eta f)\eta(x)\right| \leq C\varphi_1(x)\,\|f\|_{\varphi_1}\,\bar\alpha^n. \tag{10}$$

We consider the kernels R_1, R_2 and R_3 defined as

$$R_1 f(x) := \mathbb{1}_{x\in E'_K} R(f\mathbb{1}_{E'_K}),\ R_2 f(x) = \mathbb{1}_{x\in E'_K} R(f\mathbb{1}_{E_K\setminus E'_K}),\ R_3 f(x) = \mathbb{1}_{x\in E_K\setminus E'_K} R(f\mathbb{1}_{E_K\setminus E'_K}),$$

so that $R = R_1 + R_2 + R_3$ and, for all $n \geq 0$,

$$R^n = R_1^n + \sum_{k=1}^{n-1} R_1^{k-1} R_2 R_3^{n-k} + R_3^n. \tag{11}$$

Since, for all $f \in \mathscr{B}_b(\varphi_1^K \mathbb{1}_{E'_K})$ and all $x \in E'_K$, $R^n f(x) = R_1^n f(x)$, we have, according to (10) and for some constant $C > 0$ that may change from line to line, for all $x \in E_K$,

$$(r_{\varphi_1^K}(R))^{-n} R_1^n(\varphi_1)(x) \leq C\varphi_1(x). \tag{12}$$

Since $R_2\varphi_1 \leq R\varphi_1$, we also deduce that, for all $x \in E_K$,

$$R_2\varphi_1(x) \leq C\varphi_1(x). \tag{13}$$

Finally, since $K \subset E'_K$, we deduce that, for all $x \in E_K$ and by (A2),

$$R_3\varphi_1(x) \leq \theta_1\varphi_1(x). \tag{14}$$

Using inequalities (12), (13) and (14), together with the decomposition (11) and the fact that $\theta_1 < r_{\varphi_1^K}(R)$, we deduce that, for all $x \in E_K$ and all $n \geq 0$,

$$(r_{\varphi_1^K}(R))^{-n} R^n \varphi_1(x) \leq C\varphi_1(x).$$

In particular, since on the one hand φ_1 is bounded on $K \subset E_K$ and larger than $\mathbb{1}_{E_K}$, and since on the other hand $R^n \mathbb{1}_{E_K}(x) = P^n \mathbb{1}_E(x)$ for all $x \in E_K$, we deduce that

$$\sup_{n\geq 0} \sup_{x\in K} (r_{\varphi_1^K}(R))^{-n} P^n \mathbb{1}_E(x) < +\infty. \tag{15}$$

Integrating (10) with respect to ν (from (A1)) and letting $n \to +\infty$ entails that

$$\lim_{n\to+\infty} (r_{\varphi_1^K}(R))^{-n} \nu R^n \mathbb{1}_{E_K} = \nu_s(\eta)\nu(\eta) > 0,$$

and hence, using the fact that $r_{\varphi_1^K}(R))^{-n} \nu P^n \mathbb{1}_{E_K} = r_{\varphi_1^K}(R))^{-n} \nu R^n \mathbb{1}_{E_K} > 0$ for all $n \geq 0$, that

$$\inf_{n\geq 0} (r_{\varphi_1^K}(R))^{-n} \nu P^n \mathbb{1}_E > 0.$$

Since $\delta_x P_{n_1} \geq c_1 \nu$ for all $x \in K$ (this is (A1)), we deduce that

$$\inf_{n\geq 0} (r_{\varphi_1^K}(R))^{-n} P^n \mathbb{1}_E(x) > 0. \tag{16}$$

We deduce from (15) and (16) that (7) holds true, which concludes the proof of Proposition 6.

□

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