

# Practical criteria for $R$ -positive recurrence of unbounded semigroups

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## Abstract

The goal of this note is to show how recent results on the theory of quasi-stationary distributions allow to deduce effortlessly general criteria for the geometric convergence of normalized unbounded semigroups.

*Keywords:*  $R$ -positivity; quasi-stationary distributions; mixing properties; Foster-Lyapunov criteria

## 1 Introduction

Let  $E$  be a measurable space and  $(P_n, n \in \mathbb{Z}_+)$  be a positive semigroup on the set of bounded measurable functions on  $E$ . In the case where  $P_1$  is a bounded operator, one can define the dual action of  $(P_n, n \in \mathbb{Z}_+)$  on the set of probability measures on  $E$  as

$$\mu P_n f = \int_E P_n f(x) \mu(dx), \quad (1.1)$$

for all  $f$  bounded measurable and all probability measures  $\mu$  on  $E$ . In this case, the authors provided in [9] general sufficient conditions ensuring the convergence in total variation of the normalized semigroup  $\frac{\mu P_n}{\mu P_n \mathbb{1}}$ , where  $\mathbb{1}$  is the constant function equal to 1 on  $E$ , to a so-called *quasi-stationary* probability measure, with a speed bounded by  $C \alpha^n \frac{\mu(\varphi_1)}{\mu(\varphi_2)}$  for some  $\alpha \in (0, 1)$  and appropriate functions  $\varphi_1$  and  $\varphi_2$ . This result can be seen as an extension to bounded non-conservative semigroups of criteria of convergence for semigroups associated to Markov processes (in particular, Harris theorem and all its extensions based on Doeblin's conditions and Foster-Lyapunov criteria, see e.g. [25, 15]) and as a practical alternative to  $R$ -recurrent Markov chains theory [29, 27, 26]. In particular, it provides an alternative to spectral theoretic results dealing with existence of eigenfunctions and convergence to them (e.g. Krein-Rutman theorem, spectral theory of symmetric operators, or the theorem of convergence of normalized semigroups of Birkhoff [7] and its extensions).

The goal of the present note is to show how the results of [9] allow to deduce effortlessly general criteria for the geometric convergence of normalized semigroups when  $P_1$  is unbounded. This natural extension provides practical criteria for the  $R$ -positive recurrence of unbounded semigroups as developed in [27, Section 6.2] and [26]. It has applications to penalized Markov processes [13, 14], to the study of the long time behaviour of Markov branching processes (see for instance [18, 19, 20, 6, 21, 10, 5, 3, 4]), of non-conservative PDEs (see e.g. [1, 2] and references therein) and of measure-valued Pólya processes and reinforced processes [23].

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We shall consider cases where there exists a measurable (possibly unbounded) function  $\psi_1 : E \rightarrow (0, +\infty)$  such that  $P_1\psi_1 \leq c\psi_1$  for some constant  $c$ , so that the right and left action of  $P_n$  in (1.1) are defined naturally for all measurable  $f : E \rightarrow \mathbb{R}$  such that  $f/\psi_1$  is bounded and all positive measure  $\mu$  such that  $\mu(\psi_1) < +\infty$  (this corresponds to the setting described in [27, Section 6.2]). In this setting, the recent article [2] makes use of the methods developed in [8, 9] to give a necessary and sufficient condition for the existence of a positive eigenfunction  $\eta$  of  $P_1$  with eigenvalue  $\theta_0$  and the geometric convergence of  $\theta_0^{-n}\mu P_n f$  for all  $f$  and  $\mu$  such that  $f/\psi_1$  is bounded and  $\mu(\psi_1) < +\infty$ . We show below that this result can be strengthened as an immediate corollary of the results of [9] applied to the sub-Markov semigroup  $\frac{P_n(\cdot\psi_1)}{c^n\psi_1}$  for the sufficient condition, and standard results on ergodicity of Markov processes applied to a well-chosen  $h$ -transform of  $P_n$  for the necessary condition.

Section 2 is devoted to the statement and the proof of this result. We then explain in Section 3 how large classes of semigroups satisfying our hypotheses can be deduced from those studied in [9]. We focus on two applications: penalized semigroups associated to perturbed (discrete-time) dynamical systems (Subsection 3.1) and diffusion processes (Subsection 3.2).

## 2 Main result

We first introduce the assumptions on which our results are based. We state them following the same structure as Assumption (E) in [9] to emphasize their similarity.

**Condition (G).** There exist positive real constants  $\theta_1, \theta_2, c_1, c_2, c_3$ , an integer  $n_1 \geq 1$ , two functions  $\psi_1 : E \rightarrow (0, +\infty)$ ,  $\psi_2 : E \rightarrow \mathbb{R}_+$  and a probability measure  $\nu$  on a measurable subset  $K$  of  $E$  such that

(G1) (*Local Dobrushin coefficient*).  $\forall x \in K$  and all measurable  $A \subset K$ ,

$$P_{n_1}(\psi_1 \mathbb{1}_A)(x) \geq c_1 \nu(A) \psi_1(x).$$

(G2) (*Global Lyapunov criterion*). We have  $\theta_1 < \theta_2$  and

$$\begin{aligned} \inf_{x \in K} \psi_2(x)/\psi_1(x) &> 0, \quad \sup_{x \in E} \psi_2(x)/\psi_1(x) \leq 1, \\ P_1\psi_1(x) &\leq \theta_1\psi_1(x) + c_2 \mathbb{1}_K(x)\psi_1(x), \quad \forall x \in E \\ P_1\psi_2(x) &\geq \theta_2\psi_2(x), \quad \forall x \in E. \end{aligned}$$

(G3) (*Local Harnack inequality*). We have

$$\sup_{n \in \mathbb{Z}_+} \frac{\sup_{y \in K} P_n \psi_1(y)/\psi_1(y)}{\inf_{y \in K} P_n \psi_1(y)/\psi_1(y)} \leq c_3$$

(G4) (*Aperiodicity*). For all  $x \in K$ , there exists  $n_4(x)$  such that for all  $n \geq n_4(x)$ ,

$$P_n(\mathbb{1}_K \psi_1) > 0.$$

In the following theorem, we consider the Banach space

$$L^\infty(\psi_1) = \{f : E \rightarrow \mathbb{R} \text{ measurable, s.t. } f/\psi_1 \text{ is bounded}\},$$

endowed with the norm  $\|f\|_{L^\infty(\psi_1)} := \|f/\psi_1\|_\infty$ .

**Theorem 2.1.** *Assume that Condition (G) holds true. Then there exists a positive measure  $\nu_P$  on  $E$  such that  $\nu_P(\psi_1) = 1$  and  $\nu_P(\psi_2) > 0$ , and two constants  $C < +\infty$  and  $\alpha \in (0, 1)$  such that, for all measurable functions  $f : E \rightarrow \mathbb{R}$  satisfying  $|f| \leq \psi_1$  and all positive measure  $\mu$  on  $E$  such that  $\mu(\psi_1) < +\infty$  and  $\mu(\psi_2) > 0$ ,*

$$\left| \frac{\mu P_n f}{\mu P_n \psi_1} - \nu_P(f) \right| \leq C \alpha^n \frac{\mu(\psi_1)}{\mu(\psi_2)}, \quad \forall n \in \mathbb{Z}_+. \quad (2.1)$$

*In addition, there exists  $\theta_0 > 0$  such that  $\nu_P P_n = \theta_0^n \nu_P$  and a function  $\eta : E \rightarrow \mathbb{R}_+$  such that  $\theta_0^{-n} P_n \psi_1$  converges uniformly and geometrically toward  $\eta$  in  $L^\infty(\psi_1)$  and such that  $P_1 \eta = \theta_0 \eta$  and  $\nu_P(\eta) = \nu_P(\psi_1) = 1$ . Moreover, there exist two constants  $C' > 0$  and  $\beta \in (0, 1)$  such that, for all measurable functions  $f : E \rightarrow \mathbb{R}$  satisfying  $|f| \leq \psi_1$  and all positive measures  $\mu$  on  $E$  such that  $\mu(\psi_1) < +\infty$ ,*

$$|\theta_0^{-n} \mu P_n f - \mu(\eta) \nu_P(f)| \leq C' \beta^n \mu(\psi_1). \quad (2.2)$$

*Remark 1.* Note that (G2) implies that  $P_n \psi_1 \leq c P_n \psi_2$  on  $K$  for all  $n \geq 0$  and some constant  $c > 0$  (see [9, Lemma 9.6]). Hence we have, for all  $x \in K$ ,

$$P_n \psi_1(x) / \psi_1(x) \leq c P_n \psi_2(x) / \psi_1(x) \leq c P_n \psi_2(x) / \psi_2(x)$$

and

$$P_n \psi_2(x) / \psi_2(x) \leq P_n \psi_1(x) / \psi_2(x) \leq \sup_K \frac{\psi_1}{\psi_2} P_n \psi_1(x) / \psi_1(x).$$

Therefore, replacing  $\psi_1$  by  $\psi_2$  in (G1) and/or (G3) give equivalent versions of Condition (G). In [2], a similar result is obtained, but with the additional assumptions that  $\psi_2 > 0$  on  $E$  and  $n_1 = 1$ . In this restricted case, one can easily check that their assumptions on the discrete-time semigroup are equivalent to ours. The fact that  $\psi_2$  can vanish allows to deal with reducible processes (see [9, Section 6]).

*Proof.* Assumption (G2) implies that  $P_1 \psi_1 \leq (\theta_1 + c_2) \psi_1$ , so that  $Q_1 f := \frac{P_1(f \psi_1)}{(\theta_1 + c_2) \psi_1}$  defines a sub-markovian kernel generating the semigroup  $(Q_n)_{n \in \mathbb{N}}$  defined by

$$Q_n(f) = \frac{P_n(f \psi_1)}{(\theta_1 + c_2)^n \psi_1}, \quad \forall n \geq 0, \|f\|_\infty \leq 1.$$

It is straightforward to check that this semigroup satisfies conditions (E1-E4) in [9] with  $\varphi_1 = 1$  and  $\varphi_2 = \psi_2 / \psi_1$ , using  $\theta_1 / (\theta_1 + c_2)$  in place of  $\theta_1$ ,  $\theta_2 / (\theta_1 + c_2)$  in place of  $\theta_2$  and  $c_1 / (\theta_1 + c_2)^{n_1}$  in place of  $c_1$ . Using Theorem 2.1 in this reference applied to  $Q_n$ , we deduce that there exist constants  $C > 0$ ,  $\alpha \in (0, 1)$  and a probability measure  $\nu_{QSD}$  on  $E$  such that, for all bounded measurable functions  $g : E \rightarrow \mathbb{R}$  and all probability measures  $\nu$  such that  $\nu(\varphi_2) > 0$ ,

$$\left| \frac{\nu Q_n g}{\nu Q_n \mathbb{1}} - \nu_{QSD}(g) \right| \leq C \alpha^n \frac{\|g\|_\infty}{\nu(\varphi_2)}.$$

Setting  $\nu_P(dx) = \frac{1}{\psi_1(x)} \nu_{QSD}(dx)$ ,  $\mu(dx) = \frac{1}{\psi_1(x)} \nu(dx)$  and  $f = g \psi_1$ , one obtains (2.1). Similarly, Theorem 2.5 of [9] for  $Q_n$  states that there exist  $\theta_Q > 0$  such that  $\nu_{QSD} Q_n = \theta_Q^n \nu_{QSD}$  and a function  $\eta_Q : E \rightarrow \mathbb{R}_+$  such that  $\theta_Q^{-n} Q_n \mathbb{1}$  converges uniformly and geometrically toward  $\eta_Q$  in  $L^\infty$  and such that  $Q_1 \eta_Q = \theta_Q \eta_Q$ . Setting  $\eta = \eta_Q \psi_1$  and  $\theta_0 = \theta_Q (\theta_1 + c_2)$  gives the result on geometric convergence of  $\theta_0^{-n} P_n \psi_1$  to  $\eta$  in  $L^\infty(\Psi_1)$ .

It remains to prove (2.2). Note that it is sufficient to prove it for any  $\mu = \delta_x$ . If  $\eta(x) = 0$ , this is implied by the above geometric convergence. If  $\eta(x) > 0$ , then  $\eta_Q(x) > 0$  and the convergence of [9, Theorem 2.7] applied to  $Q_n$  implies that there exists  $C' < +\infty$  and  $\tilde{\alpha} \in (0, 1)$  such that, for all measurable  $g : E \rightarrow \mathbb{R}$  satisfying  $|g| \leq 1/\eta_Q$ ,

$$\left| \theta_Q^{-n} \frac{Q_n(g\eta_Q)(x)}{\eta_Q(x)} - \nu_{QSD}(g\eta_Q) \right| \leq C' \tilde{\alpha}^n \frac{1}{\eta_Q(x)}.$$

Multiplying both sides by  $\eta_Q(x)\psi_1(x)$  and setting  $f = g\eta_Q\psi_1$  ends the proof of (2.2).  $\square$

*Remark 2.* The elementary method consisting in studying the sub-Markov semi-group  $(Q_n)$  instead of  $(P_n)$  as done in the above proof is not particular to our assumptions. It can also be used to derive immediately sufficient criteria for the convergence of unbounded semi-groups from the abundant theory of sub-Markovian semi-groups, as developed for instance in [12, 11, 30, 16, 22, 17]. Note that a similar approach has been used in [5] to describe the asymptotic behaviour of the growth-fragmentation equation using Krein-Rutman theorem and other criteria for  $R$ -positivity.

Whether Assumption (G) is necessary for (2.1) is still an open problem up to our knowledge. However, if one assumes that there exists a positive eigenfunction  $\eta$  such that (2.2) holds true, then one recovers easily Assumption (G) by applying the classical counterpart of Forster-Lyapunov criteria for conservative semigroups. Here, the conservative semigroup is the one associated to the  $h$ -transform of  $P_n$  defined by  $R_n f := \frac{\theta_0^{-n}}{\eta} P_n(\eta f)$  (which is called  $Q$ -process in the sub-Markovian case, cf. e.g. [24]). The only difficulty in the proof of the following theorem is that  $\eta$  may vanish on some subset of  $E$ .

**Theorem 2.2.** *Assume that there exist a positive function  $\psi : E \rightarrow (0, +\infty)$  and a non-negative eigenfunction  $\eta \in L^\infty(\psi)$  of  $P_1$  for the eigenvalue  $\theta_0 > 0$ , such that*

$$|\theta_0^{-n} P_n f(x) - \eta(x)\nu_P(f)| \leq \zeta_n \psi(x) \tag{2.3}$$

*is satisfied for all  $x \in E$  and all measurable functions  $f : E \rightarrow \mathbb{R}$  such that  $|f| \leq \psi$ , where  $(\zeta_n)_{n \geq 0}$  is some positive sequence converging to 0. Then Assumption (G) is satisfied with  $\psi_2 = \eta$  and with some function  $\psi_1 \in L^\infty(\psi)$  such that  $\psi \in L^\infty(\psi_1)$ .*

*Remark 3.* A similar partial counterpart to Theorem 2.2 was proven in [2], where the authors assume that  $\zeta_n$  is geometrically decreasing, that  $\eta$  is positive and use the approach of [8] to conclude.

*Proof.* We define  $E' = \{x \in E, \eta(x) > 0\}$  and introduce the conservative semigroup  $R$  on functions  $g : E' \rightarrow \mathbb{R}$  such that  $|g(x)| \leq \psi(x)/\eta(x)$  defined by

$$R_n g(x) = \frac{\theta_0^{-n}}{\eta(x)} P_n(\eta g)(x), \quad \forall x \in E' \text{ and } n \geq 0.$$

Applying (2.3) to  $f = g\eta$  and setting  $\nu_R(dx) = \eta(x)\nu_P(dx)$ , we deduce that, for all  $x \in E'$  and all measurable function  $g : E' \rightarrow \mathbb{R}$  such that  $|g| \leq \psi/\eta$

$$|R_n g(x) - \nu_R(g)| \leq \zeta_n \frac{\psi(x)}{\eta(x)}.$$

This is the classical  $V$ -uniform ergodicity condition (with  $V = \psi/\eta$ ), for which necessary and sufficient conditions are well-known. First, it implies  $V$ -uniform geometric ergodicity, i.e. one can

replace  $\zeta_n$  by  $C\beta^n$  for some  $C > 0, \beta \in (0, 1)$  in the above equation (see for instance Proposition 15.2.3 in [15]). Second, we deduce using for example Theorem 15.2.4(b) in [15] that, for any integer  $m$  such that  $C^{1/m}\beta < 1$  and any  $\lambda, \rho$  such that  $C^{1/m}\beta \leq \lambda < \rho < 1$ , there exist  $d, C_R < +\infty$  such that

$$R_1 V_0(x) \leq \rho V_0(x) + C_R \mathbb{1}_K(x), \quad \forall x \in E', \quad (2.4)$$

with

$$V_0 = \sum_{k=0}^{m-1} \lambda^{-k} R_k \left( \frac{\psi}{\eta} \right)$$

and  $K := \{\psi/\eta \leq d\} \cap E'$  is an accessible small set for  $R$ . This last property means that there exists a probability measure  $\nu_R$  on  $E'$  and a constant  $c_R > 0$  such that, for all  $A \subset K$  measurable,

$$R_{n'_1} \mathbb{1}_A(x) \geq c_R \nu_R(A), \quad \forall x \in K.$$

for some constant integer  $n'_1 \geq 1$ . Since  $K$  is accessible, there exists  $n''_1 \geq 0$  such that  $a := \nu_R R_{n''_1} \mathbb{1}_K > 0$ . Setting  $n_1 = n'_1 + n''_1$ , it then follows that

$$P_{n_1}(\psi \mathbb{1}_A)(x) \geq c_R \theta_0^{n_1} \eta(x) \nu_R R_{n''_1} \left( \mathbb{1}_K \mathbb{1}_A \frac{\psi}{\eta} \right), \quad \forall x \in K.$$

Due to the definition of  $K$ , we deduce that (G1) holds true with  $c_1 = ac_R \theta_0^{n_1} / d$  and the probability measure  $\nu(dx) = \frac{\psi(x)}{a\eta(x)} \mathbb{1}_K(x) (\nu_R R_{n''_1})(dx)$ .

Defining  $\psi_1 = \eta V_0$ , we also deduce from (2.4) that,

$$P_1 \psi_1(x) \leq \theta_0 \rho \psi_1(x) + C_R \mathbb{1}_K(x) \eta(x) \leq \theta_0 \rho \psi_1(x) + \frac{C_R}{\|\eta\|_{L^\infty(\psi)}} \mathbb{1}_K(x) \psi_1(x), \quad \forall x \in E'.$$

In view of the definition of  $V_0(x)$  for all  $x \in E'$ , we have

$$\psi_1(x) = \sum_{k=0}^{m-1} (\lambda \theta_0)^{-k} P_k \psi(x),$$

which also makes sense for  $x \in E \setminus E'$ . For such an  $x$ , we deduce from (2.3) that  $P_n \psi(x) \leq \zeta_n \theta_0^n \psi(x)$ . Without loss of generality, increasing  $m, \lambda$  and  $\rho$  if necessary, we can assume that  $\zeta_m^{1/m} \leq \lambda < \rho < 1$ . Then,

$$P_1 \psi_1(x) = \lambda \theta_0 \psi_1(x) - \lambda \theta_0 \psi(x) + (\lambda \theta_0)^{1-m} P_m \psi \leq \lambda \theta_0 \psi_1(x), \quad \forall x \in E \setminus E'.$$

Hence, we have checked that  $P_1 \psi_1 \leq \theta_0 \rho \psi_1 + c_2 \mathbb{1}_K \psi_1$  on  $E$  for some constants  $\rho < 1$  and  $c_2 < +\infty$ . Since  $P_1 \eta = \theta_0 \eta$ , the proof of (G2) is completed. Note also that  $\psi \leq \psi_1$  and the fact that  $\psi_1 \in L^\infty(\psi)$  follows from the inequality  $P_n \psi_1 \leq A_n \psi_1$  for some constant  $A_n$ , which is an immediate consequence of (2.3) and the fact that  $\eta \in L^\infty(\psi_1)$ .

Thanks to Remark 1, it is sufficient to check (G3) with  $\psi_2 = \eta$  instead of  $\psi_1$ . Since  $\eta$  is an eigenfunction of  $P_1$ , (G3) is trivial.

Since  $K \subset E'$ , it follows from (2.3) that, for all  $x \in K$ ,  $\theta_0^{-n} P_n(\mathbb{1}_K \psi_1)(x)$  converges as  $n \rightarrow +\infty$  to  $\eta(x) \nu_P(\mathbb{1}_K \psi_1) > 0$ . Hence (G4) is clear.  $\square$

For continuous time semigroups  $(P_t)_{t \in [0, +\infty)}$ , the conclusions of Theorem 2.1 can be easily deduced from properties on the discrete skeleton  $(P_{nt_0})_{n \in \mathbb{N}}$  (similar properties were already observed in Theorem 5 of [29] and in [9]). In the following result, the function  $\eta$  and the positive measure  $\nu_P$  are the one of Theorem 2.1 applied to the discrete skeleton  $(P_{nt_0})_{n \in \mathbb{N}}$ .

**Corollary 2.3.** *Let  $(P_t)_{t \in [0, +\infty)}$  be a continuous time semigroup. Assume that there exists  $t_0 > 0$  such that  $(P_{nt_0})_{n \in \mathbb{N}}$  satisfies Assumption (G),  $\left(\frac{P_t \psi_1}{\psi_1}\right)_{t \in [0, t_0]}$  is upper bounded by a constant  $\bar{c} > 0$  and  $\left(\frac{P_t \psi_2}{\psi_2}\right)_{t \in [0, t_0]}$  is lower bounded by a constant  $\underline{c} > 0$ . Then there exist some constants  $C'' > 0$  and  $\gamma > 0$  such that, for all measurable functions  $f : E \rightarrow \mathbb{R}$  satisfying  $|f| \leq \psi_1$  and all positive measure  $\mu$  on  $E$  such that  $\mu(\psi_1) < +\infty$  and  $\mu(\psi_2) > 0$ ,*

$$\left| \frac{\mu P_t f}{\mu P_t \psi_1} - \nu_P(f) \right| \leq C'' e^{-\gamma t} \frac{\mu(\psi_1)}{\mu(\psi_2)}, \quad \forall t \in [0, +\infty), \quad (2.5)$$

*In addition, there exists  $\lambda_0 \in \mathbb{R}$  such that  $\nu_P P_t = e^{\lambda_0 t} \nu_P$  for all  $t \geq 0$ , and  $e^{-\lambda_0 t} P_t \psi_1$  converges uniformly and exponentially toward  $\eta$  in  $L^\infty(\psi_1)$  when  $t \rightarrow +\infty$ . Moreover, there exist some constants  $C''' > 0$  and  $\gamma' > 0$  such that, for all measurable functions  $f : E \rightarrow \mathbb{R}$  satisfying  $|f| \leq \psi_1$  and all positive measures  $\mu$  on  $E$  such that  $\mu(\psi_1) < +\infty$ ,*

$$\left| e^{-\lambda_0 t} \mu P_t f - \mu(\eta) \nu_P(f) \right| \leq C''' e^{-\gamma' t} \mu(\psi_1), \quad \forall t \in [0, +\infty). \quad (2.6)$$

*Proof.* Assuming without loss of generality that  $t_0 = 1$  and applying (2.1) to  $\mu P_t$ , where  $t \in [0, 1]$ , and  $f$  such that  $\mu(\psi_1) < +\infty$  and  $|f| \leq \psi_1$ , one deduces that

$$\left| \frac{\mu P_{t+n} f}{\mu P_{t+n} \psi_1} - \nu_P(f) \right| \leq C \alpha^n \frac{\mu P_t \psi_1}{\mu P_t \psi_2} \leq \frac{C \bar{c}}{\alpha \underline{c}} \alpha^{n+t} \frac{\mu(\psi_1)}{\mu(\psi_2)},$$

which implies (2.5). Then, applying this inequality to  $\mu = \nu_P$  and letting  $n$  go to infinity shows that  $\nu_P P_t f / \nu_P P_t \psi_1 = \nu_P f$  for all  $t \geq 0$ . Choosing  $f = P_s \psi_1$  entails  $\nu_P P_{t+s} \psi_1 = \nu_P P_t \psi_1 \nu_P P_s \psi_1$  for all  $s, t \geq 0$ , and hence  $\nu_P P_t \psi_1 = e^{\lambda_0 t} \nu_P \psi_1$  for all  $t \geq 0$  for some constant  $\lambda_0 \in \mathbb{R}$  (note that  $\theta_0 = e^{\lambda_0}$ ).

Similarly, inequality (2.2) applied to  $\mu = \delta_x P_t$  and  $f = \psi_1$  on the one hand and to  $\mu = \delta_x$  and  $f = P_t \psi_1$  on the other hand implies that  $P_t \eta(x) = \eta(x) \nu_P(P_t \psi_1) = e^{\lambda_0 t} \eta(x)$  for all  $t \geq 0$ . Applying again (2.2) to  $\mu = \delta_x P_t$  entails that

$$\left| \theta_0^{-n} P_{t+n} f(x) - P_t \eta(x) \nu_P(f) \right| \leq C' \beta^n P_t \psi_1(x) \leq \frac{C' \bar{c}}{\beta} \beta^{n+t} \psi_1(x).$$

In particular, for all  $t \geq 0$ ,

$$\left| e^{-\lambda_0 t} P_t f(x) - \eta(x) \nu_P(f) \right| \leq \frac{C' \bar{c}}{\beta} \beta^t \psi_1(x)$$

and  $e^{-\lambda_0 t} P_t \psi_1$  converges geometrically to  $\eta$  in  $L^\infty(\psi_1)$ . This concludes the proof of Corollary 2.3  $\square$

### 3 Some applications

Given a positive semigroup  $P$  acting on measurable functions on  $E$ , one can try to directly check Assumption (G) by finding appropriate functions  $\psi_1$  and  $\psi_2$ . Another natural and equivalent strategy is to find a function  $\psi$  such that the semigroup defined by  $Q_n f = \frac{P_n(\psi f)}{c^n \psi}$  is sub-Markovian and check that it satisfies Assumption (E) of [8]. The main advantage of this last approach is that  $Q$  has a probabilistic interpretation as the semigroup of a sub-Markov process. As such, one can apply

all the criteria developed in the above mentioned reference and, more generally, use the intuitions and toolboxes of the theory of stochastic processes. Since both approaches are equivalent, this is rather a question of taste.

In Subsection 3.1, we consider the case of a penalized perturbed dynamical system and check Assumption (G) directly. In subsection 3.2, we consider the case of a penalized diffusion processes and check Assumption (E).

### 3.1 Perturbed dynamical systems

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a locally bounded measurable function and consider the perturbed dynamical system  $X_{n+1} = f(X_n) + \xi_n$  with  $(\xi_i)_{i \in \mathbb{Z}_+}$  i.i.d. non-degenerate Gaussian random variables. We are interested in the asymptotic behaviour of the associated Feynman-Kac semigroup

$$P_n f(x) = \mathbb{E}_x \left( \prod_{k=1}^n G(X_k) \mathbb{1}_{X_k \in E} f(X_n) \right),$$

where  $E$  is a measurable subset of  $\mathbb{R}^d$  with positive Lebesgue measure and  $G : E \rightarrow (0, +\infty)$  is a measurable function.

**Proposition 3.1.** *Assume that  $1/G$  is locally bounded,  $G(x) \leq C \exp(|x|)$  for all  $x \in E$  and some constant  $C > 0$  and there exists  $p > 1$  such that  $|x| - p|f(x)| \rightarrow +\infty$  when  $|x| \rightarrow +\infty$ , then the semigroup  $(P_n)_{n \in \mathbb{N}}$  satisfies Assumption (G).*

*Proof.* One easily checks that  $\psi_1(x) = \exp(a|x|)$ , where  $a > 0$  is such that  $1/a < p - 1$ , satisfies

$$P_1 \psi_1(x) \leq C \mathbb{E} \left( e^{(1+a)|f(x)+\xi_1|} \right) \leq C' \psi_1(x) \exp(-a(|x| - p|f(x)|)), \quad (3.1)$$

where  $C' = C \mathbb{E} e^{(1+a)|\xi_1|}$ . Now, assume without loss of generality that  $B(0, 1) \cap E$  has positive Lebesgue measure and set  $\theta_2 := \inf_{x \in B(0, 1) \cap E} P_1 \mathbb{1}_{B(0, 1) \cap E}(x) / 2$ , which is clearly positive. It then follows from Markov's property that

$$\theta_2^{-n} \inf_{x \in B(0, 1) \cap E} P_n \mathbb{1}_{B(0, 1) \cap E}(x) \geq \theta_2^{-n} \inf_{x \in B(0, 1) \cap E} \mathbb{E}_x \left[ \prod_{k=1}^n G(X_k) \mathbb{1}_{B(0, 1) \cap E}(X_k) \right] \geq 2^n \rightarrow +\infty,$$

when  $n \rightarrow +\infty$ . One easily deduces that, for all  $R \geq 1$ ,  $\theta_2^{-n} \inf_{x \in B(0, R) \cap E} P_n \mathbb{1}_{B(0, 1) \cap E}(x) \rightarrow +\infty$ , and hence that  $\theta_2^{-n} \inf_{x \in B(0, R) \cap E} P_n \mathbb{1}_{B(0, R) \cap E}(x) \rightarrow +\infty$  when  $n \rightarrow +\infty$ .

We set  $\theta_1 = \theta_2 / 2$  and fix  $R \geq 1$  large enough so that  $C' e^{-a(|x| - p|f(x)|)} \leq \theta_1$  for all  $|x| \geq R$ . It then follows from (3.1) that  $P_1 \psi_1 \leq \theta_1 \psi_1 + c_2 \mathbb{1}_K \psi_1$ , where  $K := B(0, R) \cap E$ . Setting  $\psi_2(x) = \sum_{k=0}^{n_0} \theta_2^{-k} P_k \mathbb{1}_K$ , we deduce that, for all  $x \in E$ ,

$$P_1 \psi_2(x) = \sum_{k=0}^{n_0} \theta_2^{-k} P_{k+1} \mathbb{1}_K(x) = \theta_2 \psi_2(x) + \theta_2 \left[ \theta_2^{-(n_0+1)} P_{n_0+1} \mathbb{1}_K(x) - \mathbb{1}_K(x) \right] \geq \theta_2 \psi_2(x)$$

for  $n_0$  chosen large enough. Since in addition  $P_k \mathbb{1}_K \leq P_k \psi_1 \leq (\theta_1 + c_2)^k \psi_1$ ,  $\psi_2 \in L^\infty(\psi_1)$  and, for all  $x \in K$ ,  $\psi_2(x) \geq 1 \geq e^{-aR} \psi_1(x)$ . Hence, dividing  $\psi_2$  by  $\|\psi_2\|_{L^\infty(\psi_1)}$  ends the proof of (G2).

In order to prove (G1), (G3) and (G4), we follow similar arguments as for [9, Prop. 7.2]. Since the adaptation of these arguments is a bit tricky because the function  $\psi_1$  needs to be taken into account appropriately, we give the details below.

The first step consists in proving that there exists a constant  $c > 0$  such that, for all measurable  $A \subset K$ , for all  $x \in E$  and all  $y \in K$ ,

$$\frac{P_1(\psi_1 \mathbb{1}_A)(x)}{\psi_1(x)} \leq c \frac{P_1(\psi_1 \mathbb{1}_A)(y)}{\psi_1(y)}. \quad (3.2)$$

This implies easily (G1) for  $n_1 = 1$  and (G4) then follows directly from (G1) (since  $n_1 = 1$ ).

To prove (3.2), we observe that (recall that  $A \subset K = E \cap B(0, R)$ )

$$\frac{P_1(\psi_1 \mathbb{1}_A)(x)}{\psi_1(x)} \leq P_1(\psi_1 \mathbb{1}_A)(x) \leq \sup_{|z| \leq R} [G(z)\psi_1(z)] \mathbb{P}(f(x) + \xi_1 \in E \cap A \cap B(0, R)).$$

Because  $\xi_1$  is a non-degenerate gaussian random variable, it is not hard to check that there exists a constant  $C_R$  depending only on  $R$  (and not on  $x \in E$  and  $y \in K$ ) such that  $\mathbb{P}(f(x) + \xi_1 \in E \cap A \cap B(0, R)) \leq C_R \mathbb{P}(f(y) + \xi_1 \in E \cap A \cap B(0, R))$ . Therefore,

$$\frac{P_1(\psi_1 \mathbb{1}_A)(x)}{\psi_1(x)} \leq C_R \frac{\sup_{|z| \leq R} G(z)\psi_1(z)}{\inf_{|z| \leq R} G(z)} \mathbb{E}_y [G(X_1)\psi_1(X_1) \mathbb{1}_{X_1 \in E \cap A}] \leq c \frac{P_1(\psi_1 \mathbb{1}_A)(y)}{\psi_1(y)},$$

where  $c = C_R e^{aR} \sup_{|z| \leq R} G(z)\psi_1(z) / \inf_{|z| \leq R} G(z)$ . Hence (3.2) is proved.

Next, we observe that the Markov property combined with (G2) implies that, for all  $x \in E$  and all  $n \geq 1$ ,

$$\mathbb{E}_x \left[ \prod_{k=1}^n G(X_k) \mathbb{1}_{X_k \in E \setminus K} \psi_1(X_n) \right] \leq (\theta_1 + c_2) \theta_1^{n-1} \psi_1(x). \quad (3.3)$$

We also have the property that there exists a constant  $c' > 0$  such that, for all  $y \in K$  and all  $0 \leq k \leq n$ ,

$$\frac{P_n \psi_1(y)}{\psi_1(y)} \geq c' \theta_2^k \frac{P_{n-k} \psi_1(y)}{\psi_1(y)}. \quad (3.4)$$

As observed in Remark 1, since we already proved (G2), the last property is equivalent to the same one with  $\psi_2$  instead of  $\psi_1$ . Since  $P_1 \psi_2 \geq \theta_2 \psi_2$  on  $K$  [du coup, ça n'itère pas bien ici... je ne retrouve plus l'argument] (3.4) is then clear.

The proof of (G3) can then be done combining the last inequalities. We first decompose  $P_n \psi_1$  depending on the value of the first return time in  $K$ : for all  $x \in E$ ,

$$\begin{aligned} P_n \psi_1(x) &= \mathbb{E}_x \left[ \prod_{k=1}^n G(X_k) \mathbb{1}_{X_k \in E \setminus K} \psi_1(X_n) \right] + \sum_{\ell=1}^n \mathbb{E}_x \left[ \prod_{k=1}^{\ell-1} G(X_k) \mathbb{1}_{X_k \in E \setminus K} G(X_\ell) \mathbb{1}_{X_\ell \in K} P_{n-\ell} \psi_1(X_\ell) \right] \\ &\leq (\theta_1 + c_2) \theta_1^{n-1} \psi_1(x) + \sum_{\ell=1}^n \mathbb{E}_x \left[ \prod_{k=1}^{\ell-1} G(X_k) \mathbb{1}_{X_k \in E \setminus K} \mathbb{E}_{X_{\ell-1}} [G(X_1) \mathbb{1}_{X_1 \in K} P_{n-\ell} \psi_1(X_1)] \right], \end{aligned}$$

where we used (3.3) and Markov's property in the second line. We then proceed by using (3.2) for some fixed  $y \in K$  first, (3.3) next, and finally (3.4) twice:

$$\begin{aligned} \frac{P_n \psi_1(x)}{\psi_1(x)} &\leq (\theta_1 + c_2) \theta_1^{n-1} + \frac{c}{\psi_1(x)} \sum_{\ell=1}^n \mathbb{E}_x \left[ \prod_{k=1}^{\ell-1} G(X_k) \mathbb{1}_{X_k \in E \setminus K} \psi_1(X_{k-1}) \right] \frac{\mathbb{E}_y [G(X_1) \mathbb{1}_{X_1 \in K} P_{n-\ell} \psi_1(X_1)]}{\psi_1(y)} \\ &\leq \frac{\theta_1 + c_2}{\theta_1} \theta_1^n + \frac{c(\theta_1 + c_2)}{\theta_1} \sum_{\ell=1}^n \theta_1^{\ell-1} \frac{P_{n-\ell+1} \psi_1(y)}{\psi_1(y)} \\ &\leq \left[ \frac{\theta_1 + c_2}{c' \theta_1} \left( \frac{\theta_1}{\theta_2} \right)^n + \frac{c(\theta_1 + c_2)}{c' \theta_1} \sum_{\ell=1}^n \left( \frac{\theta_1}{\theta_2} \right)^{\ell-1} \right] \frac{P_n \psi_1(y)}{\psi_1(y)}. \end{aligned}$$

Since the last factor is bounded in  $n$ , this ends the proof of Proposition 3.1.  $\square$



### 3.2 Diffusion processes

Let  $(X_t)_{t \in [0, +\infty)}$  be solution to the SDE

$$dX_t = dB_t + b(X_t) dt, \quad X_0 \in (0, +\infty)^d, \quad (3.5)$$

where  $B$  is a standard  $d$ -dimensional Brownian motion and  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is locally Hölder. Let  $r : (0, +\infty)^d \rightarrow \mathbb{R}$  be locally bounded and consider the semigroup  $(P_t)_{t \in [0, +\infty)}$  defined by

$$P_t f(x) = \mathbb{E}_x \left( e^{\int_0^t r(X_u) du} f(X_t) \mathbb{1}_{X_s \in (0, +\infty)^d, \forall s \in [0, t]} \right). \quad (3.6)$$

The term  $\mathbb{1}_{X_s \in (0, +\infty)^d, \forall s \in [0, t]}$  above corresponds to a killing at the boundary of the domain  $(0, +\infty)^d$ . Note that the solution to (3.5) may explode in finite time if  $b$  does not satisfy the linear growth condition. However, we assume by convention that  $X_t \notin (0, +\infty)^d$  after the explosion time, so that (3.6) makes sense. We refer to [9, Sections 4.1 and 12.1] for the precise construction of the process.

One motivation for the study of this semigroup comes from the Feynman-Kac formula. Indeed, when the coefficients are smooth enough (see for instance [28, Section 1.3.3]), this semigroup is solution to the Cauchy linear parabolic partial differential equation

$$\begin{aligned} r v - \frac{\partial v}{\partial t} + \mathcal{L} v &= 0, \text{ on } [0, +\infty) \times (0, +\infty)^d \\ v(0, \cdot) &= f, \text{ on } (0, +\infty)^d, \end{aligned}$$

where  $\mathcal{L}$  is the differential operator of second order

$$\mathcal{L}\varphi(x) = \frac{1}{2} \Delta \varphi(x) + b(x) \cdot \nabla \varphi(x), \quad \forall \varphi \in C^2(\mathbb{R}^d),$$

with Dirichlet boundary conditions.

**Proposition 3.2.** *Assume that*

$$r(x) + \sum_{i=1}^d b_i(x) \xrightarrow[|x| \rightarrow \infty, x \in (0, \infty)^d]{} -\infty. \quad (3.7)$$

*Then the semigroup  $(P_t)_{t \in [0, +\infty)}$  satisfies the assumptions of Corollary 2.3.*

*Proof.* We first observe that, setting  $\psi(x) = \exp(\sum_{i=1}^n x_i)$  and  $a := d/2 + \sup_{x \in (0, \infty)^d} r(x) + \sum_{i=1}^d b_i(x)$ , we have

$$Q_t f := e^{-at} \frac{P_t(f\psi)(x)}{\psi(x)} = \mathbb{E}_x \left( e^{-\frac{d}{2}t + \sum_{i=1}^d B_t^i} e^{\int_0^t (r(X_u) + \sum_{i=1}^d b_i(X_u) - a + \frac{d}{2}) du} f(X_t) \mathbb{1}_{X_s \in (0, +\infty)^d, \forall s \in [0, t]} \right).$$

Using Girsanov's theorem, we deduce that

$$Q_t f = \mathbb{E}_x \left( e^{-\int_0^t \kappa(\bar{X}_u) du} f(\bar{X}_t) \mathbb{1}_{\bar{X}_s \in (0, +\infty)^d, \forall s \in [0, t]} \right).$$

where  $\kappa(y) = a - r(y) - \frac{d}{2} - \sum_{i=1}^d b_i(y) \geq 0$  and  $\bar{X}$  is solution to the SDE  $d\bar{X}_t = dB_t + \frac{dt}{2} + b(\bar{X}_t) dt$  with  $\bar{X}_0 = x$ .

Assumption (3.7) thus implies that the conditions of [9, Theorem 4.5] are satisfied<sup>1</sup> and hence that  $Q$  satisfies Assumption (F) therein, which implies that Assumption (E) is satisfied by the semigroup  $Q_{nt_0}$  for some  $t_0 > 0$  and some Lyapunov functions  $\varphi_1$  and  $\varphi_2$ , that  $\left(\frac{Q_t\varphi_1}{\varphi_1}\right)_{t \in [0, t_0]}$  is uniformly bounded, and that there exist a positive function  $\eta_Q \in L^\infty(\varphi_1)$  and a constant  $\lambda_0 > 0$  such that  $Q_t\eta_Q = e^{-\lambda_0 t}\eta_Q$  for all  $t \in [0, +\infty)$ .

To conclude, it remains to observe that the same procedure as the one used in the proof of Theorem 2.1 above allows to deduce from these properties that  $(P_{nt_0})_{n \geq 0}$  satisfies Assumption (G) with  $\psi_1 = \psi\varphi_1$  and  $\psi_2 = \psi\eta_Q$ . Observing also that  $\psi_2$  is the function  $\eta$  of Theorem 2.1, we deduce that  $(P_t)_{t \in [0, +\infty)}$  satisfies the assumptions of Corollary 2.3.  $\square$

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<sup>1</sup>To prove (4.12) therein, one can use the same argument as the one used in Corollary 4.3 of this reference.

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