

A probabilistic approach to Dirac concentration in nonlocal models of adaptation with several resources

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Abstract

This work is devoted to the study of scaling limits in small mutations and large time of the solutions u^ε of two deterministic models of phenotypic adaptation, where the parameter $\varepsilon > 0$ scales the size of mutations. The first model is the so-called Lotka-Volterra parabolic PDE in \mathbb{R}^d with an arbitrary number of resources and the second one is an adaptation of the first model to a finite phenotype space. The solutions of such systems typically concentrate as Dirac masses in the limit $\varepsilon \rightarrow 0$. Our main results are, in both cases, the representation of the limits of $\varepsilon \log u^\varepsilon$ as solutions of variational problems and regularity results for these limits. The method mainly relies on Feynman-Kac type representations of u^ε and Varadhan's Lemma. Our probabilistic approach applies to multi-resources situations not covered by standard analytical methods and makes the link between variational limit problems and Hamilton-Jacobi equations with irregular Hamiltonians that arise naturally from analytical methods. The finite case presents substantial difficulties since the rate function of the associated large deviation principle has non-compact level sets. In that case, we are also able to obtain uniqueness of the solution of the variational problem and of the associated differential problem which can be interpreted as a Hamilton-Jacobi equation in finite state space.

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1 Introduction

We are interested in the dynamics of a population subject to mutations and selection driven by competition for resources. Each individual in the population is characterized by a quantitative phenotypic trait $x \in \mathbb{R}$ (for example the size of individuals, their mean size at division for bacteria, their rate of nutrients intake or their efficiency in nutrients assimilation).

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The partial differential equations we study are mutation-competition models taking the form of reaction-diffusion equations, with non-local density-dependence in the growth rate and which have been studied in various contexts by a lot of authors. The general equation takes the form

$$\partial_t u^\varepsilon(t, x) = \frac{\varepsilon}{2} \Delta u^\varepsilon(t, x) + \frac{1}{\varepsilon} u^\varepsilon(t, x) R(x, v_t^\varepsilon), \quad \forall t > 0, x \in \mathbb{R}^d \quad (1.1)$$

where x corresponds to the phenotypic traits characterizing individuals, $u^\varepsilon(t, x)$ is the density of population with trait x at time t and $v_t^\varepsilon = (v_t^{1,\varepsilon}, \dots, v_t^{r,\varepsilon})$ with

$$v_t^{i,\varepsilon} = \int_{\mathbb{R}^d} \Psi_i(x) u^\varepsilon(t, x) dx, \quad 1 \leq i \leq r,$$

for some functions $\Psi_i : \mathbb{R}^d \rightarrow \mathbb{R}_+$. In (1.1), the Laplace operator models mutation and $R(x, v_t^\varepsilon)$ is the growth rate of individuals with trait x at time t . Competition occurs through the functions $v_t^{i,\varepsilon}$, which depend on u^ε . A typical example of function R is given by

$$R(x, v) = \sum_{i=1}^r \frac{c_i \Psi_i(x)}{1 + v_i} - d(x), \quad (1.2)$$

where the first term models births which occur through the consumption of r resources whose concentrations at time t are given by $c_i/(1 + v_i)$ and with a trait-dependent consumption efficiency given by the function $\Psi_i(x)$, and the second term corresponds to deaths without competition at trait-dependent rate $d(x)$. This form of the function R is relevant for populations of micro-organisms in a chemostat, and has been studied for related models in lots of works [16, 10, 8, 9].

The parameter $\varepsilon > 0$ in (1.1) introduces a scaling (in the limit $\varepsilon \rightarrow 0$) of small or rare mutations (this is the same for the Laplace operator) and of large time, which was introduced in models of adaptive dynamics in [16], and has been used since a long time to study front propagation in standard reaction-diffusion problems [20, 2, 22, 23]. In our model, the qualitative outcome is that solutions concentrate as Dirac masses, and this concentration is studied using the WKB ansatz

$$u^\varepsilon(t, x) = \exp\left(\frac{\varphi^\varepsilon(t, x)}{\varepsilon}\right)$$

in [16, 4, 35, 3, 27, 8, 30] for different particular cases of (1.1) and also in [36, 14, 24, 37, 27] for models with competitive Lotka-Volterra competition. Several of these works prove the convergence along a subsequence $(\varepsilon_k)_{k \geq 1}$ converging to 0 of φ^ε to a solution φ of the Hamilton-Jacobi problem

$$\partial_t \varphi(t, x) = R(x, v_t) + \frac{1}{2} |\nabla \varphi(t, x)|^2, \quad (1.3)$$

where v_t is expected to take the form

$$v_t^i = \int_{\mathbb{R}^d} \Psi_i(x) \mu_t(dx),$$

where μ_t is some (measure, weak) limit of $u^{\varepsilon_k}(t, x)$. Due to the fact that, under general assumptions, the total mass of the population $\int_{\mathbb{R}^d} u^\varepsilon(t, x) dx$ is uniformly bounded and bounded away from 0, the

function φ satisfies the constraint $\sup_{x \in \mathbb{R}^d} \varphi(t, x) = 0$ for all $t \geq 0$, and the measure μ_t is expected to have support in $\{\varphi(t, \cdot) = 0\}$. In addition, the measure μ_t is expected to be metastable in the sense that $R(x, v_t) \leq 0$ for all x such that $\varphi(t, x) = 0$ and $R(x, v_t) = 0$ for all x in the support of μ_t (to preserve the condition $\sup_{x \in \mathbb{R}^d} \varphi(t, x) = 0$).

However, the study of the Hamilton-Jacobi problem (1.3) with (some or all of) the previous constraints is a difficult problem. For example, uniqueness is only known in general in the case $r = 1$ [32, 31] (see also [35]). In addition, all the previous references only prove the convergence to (1.3) for $r = 1$, except in [8] where the very specific model (1.2) is studied for any values of $r \geq 2$. Yet, the case $r \geq 2$ is of particular biological interest since it is the only case where a phenomenon of diversification known as *evolutionary branching* [29, 15] can occur (see [9, Prop. 3.1]).

In this article, after describing the general model and the standing assumptions and recalling its basic properties in Section 2, we propose a different approach to study the convergence of $\varepsilon \log u^\varepsilon(t, x)$ as $\varepsilon \rightarrow 0$. This approach is based on a probabilistic interpretation of the solution of the PDE (1.1) through a Feynman-Kac formula involving some functional of the stochastic process corresponding to mutations, here Brownian motion (see Section 3). This formula suggests to use large deviations, and more specifically Varadhan's lemma, to study the convergence of $\varepsilon \log u^\varepsilon(t, x)$, giving in the limit a variational problem which takes the standard form of variational problems associated to Hamilton-Jacobi equations, in Section 4. This general idea is actually not new since it goes back to works of Freidlin [22, 23] on reaction-diffusion equations. In [21], Freidlin also studied similar questions for models close to (1.1) (with different initial conditions), but only for a single resource ($r = 1$) and under the assumption that, for all $x \in \mathbb{R}^d$, there exists a unique $a(x) \in \mathbb{R}^r$ such that $R(x, a(x)) = 0$. This condition is also assumed in the more recent works [35, 3, 27, 30]. These assumptions are not needed in our study. We also present our results in a more unified framework, avoiding in particular the use of precise properties of the heat semi-group. It is therefore easy to extend our results to other mutation operators in (1.1), as discussed in Section 5. In cases where the convergence to the Hamilton-Jacobi problem is known, we also deduce as a side result the equality between the solution to the Hamilton-Jacobi problem and its variational formulation (see Section 6). Interestingly, this result does not seem to be covered by existing general results on this topic because of the possible discontinuities of the coefficients of the Hamilton-Jacobi problem.

A natural extension is the case of finite trait space E (instead of \mathbb{R}^d), in which the large deviations scaling suggests to consider

$$\dot{u}^\varepsilon(t, i) = \sum_{j \in E} \exp\left(-\frac{\mathfrak{T}(i, j)}{\varepsilon}\right) (u^\varepsilon(t, j) - u^\varepsilon(t, i)) + \frac{1}{\varepsilon} u^\varepsilon(t, i) R(i, v_t^\varepsilon), \quad \forall t \in [0, T], \forall i \in E$$

for positive $\mathfrak{T}(i, j)$. The extension needs some care since in this case the large deviations principle for the mutation process is not standard (in particular, the rate function has non-compact level sets). This is done in Section 7. Since the variational problem is simpler in this case, we can study it in detail. In Section 8, we prove that, under assumptions ensuring that one can associate a unique metastable measure μ_t to any set of zeroes of $\varphi(t, \cdot)$, the variational problem and the associated Hamilton-Jacobi problem both have a unique solution with appropriate regularity. Hence the full family $(\varepsilon \log u^\varepsilon(t, x))_{\varepsilon > 0}$ converges to this limit. A key point consists in proving that one can

characterize any accumulation point of v_{t+s}^ε for small $s > 0$ only from the zeroes of $\varphi(t, \cdot)$.

2 Problem statement and preliminary results

We consider the following partial differential equation in $\mathbb{R}_+ \times \mathbb{R}^d$:

$$\begin{cases} \partial_t u^\varepsilon(t, x) = \frac{\varepsilon}{2} \Delta u^\varepsilon(t, x) + \frac{1}{\varepsilon} u^\varepsilon(t, x) R(x, v_t^\varepsilon), & \forall t > 0, x \in \mathbb{R}^d \\ u^\varepsilon(0, x) = \exp\left(-\frac{h_\varepsilon(x)}{\varepsilon}\right), & \forall x \in \mathbb{R}^d \end{cases} \quad (2.1)$$

with

$$v_t^\varepsilon = \left(v_t^{1,\varepsilon}, \dots, v_t^{r,\varepsilon}\right),$$

where

$$v_t^{i,\varepsilon} = \int_{\mathbb{R}^d} \Psi_i(x) u^\varepsilon(t, x) dx, \quad 1 \leq i \leq r,$$

R is a map from $\mathbb{R}^d \times \mathbb{R}^r$ to \mathbb{R} and Ψ_i and h_ε are maps from \mathbb{R}^d to \mathbb{R} .

Let us state our assumptions of R , ψ_i and h_ε .

1. Assumptions on Ψ_i

- There exist Ψ_{\min} and Ψ_{\max} , two positive real numbers such that

$$\Psi_{\min} \leq \Psi_i(x) \leq \Psi_{\max}, \quad \forall x \in \mathbb{R}^d \text{ and } \Psi_i \in W^{2,\infty}(\mathbb{R}^d), \quad \forall 1 \leq i \leq r. \quad (2.2)$$

2. Assumptions on R

- R is continuous on $\mathbb{R}^d \times \mathbb{R}^r$.
- There exists A a positive real number such that

$$-A \leq \partial_{v_i} R(x, v_1, \dots, v_r) \leq -A^{-1}, \quad \forall i \in \{1, \dots, r\}, x \in \mathbb{R}^d, v_1, \dots, v_r \in \mathbb{R}.$$

- There exist two positive constants $v_{\min} < v_{\max}$ such that

$$\min_{x \in \mathbb{R}^d} R(x, v) > 0 \text{ as soon as } \|v\|_1 < v_{\min}, \text{ and } \max_{x \in \mathbb{R}^d} R(x, v) < 0 \text{ as soon as } \|v\|_1 > v_{\max},$$

where $\|v\|_1 = |v_1| + \dots + |v_r|$.

- Let \mathcal{H} denotes the annulus $B(x, 2v_{\max}) \setminus B(x, v_{\min}/2)$ (w.t.r. to the $\|\cdot\|_1$ norm). Then

$$\sup_{v \in \mathcal{H}} \|R(\cdot, v)\|_{W^{2,\infty}} < M.$$

Note that the constant 2 in the definition of \mathcal{H} could be replaced by any constant strictly larger than 1.

3. Assumptions on h_ε

- (a) h_ε is Lipschitz-continuous on \mathbb{R}^d , uniformly with respect to $\varepsilon > 0$.
- (b) h_ε converges in $L^\infty(\mathbb{R}^d)$ as $\varepsilon \rightarrow 0$ to a function h .
- (c) For all $\varepsilon > 0$ and all $1 \leq i \leq d$,

$$v_{\min} \leq \int_{\mathbb{R}^d} \Psi_i(x) \exp\left(-\frac{h_\varepsilon(x)}{\varepsilon}\right) dx \leq v_{\max}.$$

In particular, $u_\varepsilon(0, x)$ is bounded in $L^1(\mathbb{R}^d)$.

Note that the limit h of h_ε is continuous, and hence, in order to satisfy Assumption 3c, it must satisfy $h(x) \geq 0$ for all $x \in \mathbb{R}^d$.

This type of assumptions is standard in this domain [35, 3, 27, 32], but the previous references only studied the case $r = 1$. They are nearly identical (except for h_ε and for linear or quadratic bounds at infinity, see Corollary 6.1 in Section 6) as those of [27]. The case $r \geq 2$ was only studied in [8], but for a very specific form of the function R , and in [35] but without the convergence to the Hamilton-Jacobi problem.

Now, we present some preliminary results which are needed to study the asymptotic behaviour of the solution of (2.1). The first result, Proposition 2.1, gives preliminary estimates on the solution of (2.1). The second one, Theorem 2.2, provides the existence and uniqueness of the solution of the equation. These two results are direct adaptations of the results of [35, 3, 27]. We detail the proof of the first one for sake of completeness.

Proposition 2.1. *(A priori estimates) Suppose that there exists a weak solution u^ε in $C(\mathbb{R}_+; L^1(\mathbb{R}^d))$. We have, for all positive time t and all $i \in \{1, \dots, r\}$,*

$$\begin{aligned} v_{\min} - \frac{A\varepsilon^2\Psi_{\min}}{\|\Psi_i\|_{W^{2,\infty}}} &\leq v_t^{i,\varepsilon} \leq v_{\max} + \frac{A\varepsilon^2\Psi_{\min}}{\|\Psi_i\|_{W^{2,\infty}}}, \\ v_{\min} - \frac{A\varepsilon^2\Psi_{\min}}{\inf_{1 \leq i \leq r} \|\Psi_i\|_{W^{2,\infty}}} &\leq \|v_t^\varepsilon\|_1 \leq v_{\max} + \frac{A\varepsilon^2\Psi_{\min}}{\inf_{1 \leq i \leq r} \|\Psi_i\|_{W^{2,\infty}}} \end{aligned} \quad (2.3)$$

and

$$\Psi_{\max}^{-1} \left(v_{\min} - \frac{A\varepsilon^2\Psi_{\min}}{\inf_{1 \leq i \leq r} \|\Psi_i\|_{W^{2,\infty}}} \right) \leq \int_{\mathbb{R}^d} u^\varepsilon(t, x) dx \leq \left(v_{\max} + \frac{A\varepsilon^2\Psi_{\min}}{\inf_{1 \leq i \leq r} \|\Psi_i\|_{W^{2,\infty}}} \right) \Psi_{\min}^{-1} \quad (2.4)$$

Proof. Usual localization techniques for integration by parts entails, for all $1 \leq i \leq r$,

$$\partial_t v_t^{i,\varepsilon} = \varepsilon \int_{\mathbb{R}^d} \Delta \Psi_i(x) u^\varepsilon(t, x) dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \Psi_i(x) u^\varepsilon(t, x) R(x, v_t^\varepsilon) dx.$$

Assumptions (2.2) on Ψ_i leads to

$$\begin{aligned} -\frac{\varepsilon \|\Psi_i\|_{W^{2,\infty}}}{\Psi_{\min}} \int_{\mathbb{R}^d} \Psi_i(x) u^\varepsilon(t, x) dx + \frac{1}{\varepsilon} v_t^{i,\varepsilon} \min_{x \in \mathbb{R}^d} R(x, v_t^\varepsilon) &\leq \partial_t v_t^{i,\varepsilon} \\ &\leq \frac{\varepsilon \|\Psi_i\|_{W^{2,\infty}}}{\Psi_{\min}} \int_{\mathbb{R}^d} \Psi_i(x) u^\varepsilon(t, x) dx + \frac{1}{\varepsilon} v_t^{i,\varepsilon} \max_{x \in \mathbb{R}^d} R(x, v_t^\varepsilon), \end{aligned}$$

that is

$$-\frac{\varepsilon\|\Psi_i\|_{W^{2,\infty}}}{\Psi_{\min}}v_t^{i,\varepsilon} + \frac{1}{\varepsilon}v_t^{i,\varepsilon}\min_{x\in\mathbb{R}^d}R(x,v_t^\varepsilon) \leq \partial_tv_t^{i,\varepsilon} \leq \frac{\varepsilon\|\Psi_i\|_{W^{2,\infty}}}{\Psi_{\min}}v_t^{i,\varepsilon} + \frac{1}{\varepsilon}v_t^{i,\varepsilon}\max_{x\in\mathbb{R}^d}R(x,v_t^\varepsilon).$$

Now, if $v_t^{i,\varepsilon} > v_{\max} + \frac{A\varepsilon^2\Psi_{\min}}{\|\Psi_i\|_{W^{2,\infty}}}$ (we recall that M is the upper bound on the derivatives of R w.r.t. v), then $R(x,v_t^\varepsilon) < -\frac{\varepsilon^2\Psi_{\min}}{\|\Psi_i\|_{W^{2,\infty}}}$ for all x , which means that the derivative of $v_t^{i,\varepsilon}$ becomes negative. Because of Assumption 3c, we deduce that

$$v_t^{i,\varepsilon} \leq v_{\max} + \frac{A\varepsilon^2\Psi_{\min}}{\|\Psi_i\|_{W^{2,\infty}}}.$$

The second bound is obtained similarly and the last bound follows easily from the first one in conjunction with (2.2). \square

Theorem 2.2. *For $\varepsilon > 0$ small enough, there exists a unique solution u^ε in $C^1(\mathbb{R}_+, L^1(\mathbb{R}^d))$ of (2.1).*

Proof. We do not detail the proof since it is a direct adaptation of the unidimensional case given in [3]. We simply mention that, because of (2.4),

$$\partial_t u^\varepsilon(t, x) = \varepsilon \Delta u^\varepsilon(t, x) + \frac{1}{\varepsilon} u^\varepsilon(t, x) \tilde{R}(x, v_t^\varepsilon) \quad (2.5)$$

where

$$\tilde{R}(x, v) = \begin{cases} R(x, v) & \text{if } \|v\|_1 \in [v_{\min}/2, 2v_{\max}], \\ R(x, 2v_{\max}) & \text{if } \|v\|_1 > 2v_{\max}, \\ R(x, v_{\min}/2) & \text{if } \|v\|_1 < v_{\min}/2. \end{cases}$$

Since \tilde{R} is bounded, the non-linearity in (2.5) is Lipschitz, so we can use Theorem 3.13 in [34] to obtain the existence and uniqueness of a solution in $C([0, T], L^1(\mathbb{R}^d))$. \square

Since $R(x, v_t^\varepsilon)$ is a Lipschitz function, one can actually get higher regularity from the regularizing effect of the Laplace operator. However, since we plan to extend our method to more general mutation operators, we shall only make use in the sequel of the fact that $u^\varepsilon \in C^1(\mathbb{R}_+, L^1(\mathbb{R}^d))$.

3 Feynman-Kac representation of the solution

The purpose of this section is to prove the following integral representation of the solution of (2.1) through the so-called Feynman-Kac formula.

Theorem 3.1. *(Feynman-Kac representation of the solution of (2.1)) Let u^ε be the unique weak solution of (2.1), then*

$$u^\varepsilon(t, x) = \mathbb{E}_x \left[\exp \left(-\frac{h_\varepsilon(X_t^\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon} \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right], \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad (3.1)$$

where for all $x \in \mathbb{R}^d$, \mathbb{E}_x is the expectation associated to the probability measure \mathbb{P}_x , under which $X_0^\varepsilon = x$ almost surely and the process $B_t = (X_t^\varepsilon - x)/\sqrt{\varepsilon}$ is a standard Brownian motion in \mathbb{R}^d .

Before proving this result, we recall usual notions of weak solutions to problem (2.1) (cf. e.g. [33, 28, 17, 34]). We say that a function u in $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ is a mild solution of problem (2.1) if it satisfies the following integral equation

$$u(t, x) = P_t^\varepsilon g_\varepsilon(x) + \frac{1}{\varepsilon} \int_0^t P_{t-s}^\varepsilon (u(s, x) R(x, v_s^\varepsilon)) ds, \quad (3.2)$$

where $g_\varepsilon(x) = \exp(-h_\varepsilon(x)/\varepsilon)$ and $(P_t^\varepsilon)_{t \in \mathbb{R}_+}$ is the standard heat semi-group defined on $L^1(\mathbb{R}^d)$ by

$$P_t^\varepsilon f(x) = \int_{\mathbb{R}^d} \frac{1}{(2\pi t \varepsilon)^{n/2}} e^{-\frac{\|x-y\|^2}{2t\varepsilon}} f(y) dy, \quad \forall f \in L^1(\mathbb{R}^d).$$

We also say that a function u in $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ is a weak solution of problem (2.1) if for any compactly supported test function φ of $C^\infty([0, \infty) \times \mathbb{R}^d)$, we have

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} u(t, x) \left(-\partial_t \varphi(t, x) - \frac{\varepsilon}{2} \Delta \varphi(t, x) \right) dx dt = \frac{1}{\varepsilon} \int_{\mathbb{R}_+ \times \mathbb{R}^d} u(t, x) R(x, v_t^\varepsilon) \varphi(t, x) dx dt + \int_{\mathbb{R}^d} g_\varepsilon(x) \varphi(0, x) dx.$$

We point out that the notion of weak solution given above is the one for which existence and uniqueness hold in Theorem 2.2.

Lemma 3.2. *Let \bar{u}^ε be defined as the right-hand-side of (3.1). The function \bar{u}^ε belongs to $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ and is a mild solution of problem (2.1).*

Proof. Let t and h be two positive real numbers.

$$\begin{aligned} & \| \bar{u}^\varepsilon(t+h, \cdot) - \bar{u}^\varepsilon(t, \cdot) \|_{L^1(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d} \left| \mathbb{E}_x \left[g_\varepsilon(X_{t+h}^\varepsilon) \exp \left(\frac{1}{\varepsilon} \int_0^t R(X_s^\varepsilon, v_{t+h-s}^\varepsilon) ds \right) \exp \left(\frac{1}{\varepsilon} \int_0^h R(X_{t+s}^\varepsilon, v_{h-s}^\varepsilon) ds \right) \right. \right. \\ & \quad \left. \left. - g_\varepsilon(X_t^\varepsilon) \exp \left(\frac{1}{\varepsilon} \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right] \right| dx \\ &\leq \int_{\mathbb{R}^d} \mathbb{E}_x \left[\exp \left(\frac{1}{\varepsilon} \int_0^t R(X_s^\varepsilon, v_{t+h-s}^\varepsilon) ds \right) \left| \exp \left(\frac{1}{\varepsilon} \int_0^h R(X_{t+s}^\varepsilon, v_{h-s}^\varepsilon) ds \right) g_\varepsilon(X_{t+h}^\varepsilon) - g_\varepsilon(X_t^\varepsilon) \right| \right] dx \\ & \quad + \int_{\mathbb{R}^d} \mathbb{E}_x \left[g_\varepsilon(X_t^\varepsilon) \left| \exp \left(\frac{1}{\varepsilon} \int_0^t R(X_s^\varepsilon, v_{t+h-s}^\varepsilon) ds \right) - \exp \left(\frac{1}{\varepsilon} \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right| \right] dx. \end{aligned}$$

Using to Hypothesis (2d), we get

$$\begin{aligned} \| \bar{u}^\varepsilon(t+h, \cdot) - \bar{u}^\varepsilon(t, \cdot) \|_{L^1(\mathbb{R}^d)} &\leq \int_{\mathbb{R}^d} e^{\frac{Mt}{\varepsilon}} \left\{ \mathbb{E}_x \left[g_\varepsilon(X_{t+h}^\varepsilon) \left(e^{\frac{Mh}{\varepsilon}} - 1 \right) \right] + \mathbb{E}_x \left[|g_\varepsilon(X_{t+h}^\varepsilon) - g_\varepsilon(X_t^\varepsilon)| \right] \right\} dx \\ & \quad + \frac{M}{\varepsilon} e^{\frac{Mt}{\varepsilon}} \int_{\mathbb{R}^d} \mathbb{E}_x [g_\varepsilon(X_t^\varepsilon)] \int_0^t |v_{t+h-s}^\varepsilon - v_{t-s}^\varepsilon| ds dx. \end{aligned}$$

Since $(P_t)_{t \in \mathbb{R}_+}$ preserves the $L^1(\mathbb{R}^d)$ norm, we obtain

$$\begin{aligned} \| \bar{u}^\varepsilon(t+h, \cdot) - \bar{u}^\varepsilon(t, \cdot) \|_{L^1(\mathbb{R}^d)} &\leq \|g_\varepsilon\|_{L^1(\mathbb{R}^d)} e^{\frac{Mt}{\varepsilon}} \left(e^{\frac{Mh}{\varepsilon}} - 1 + \frac{M}{\varepsilon} \int_0^t |v_{s+h}^\varepsilon - v_s^\varepsilon| ds \right) \\ & \quad + e^{\frac{Mt}{\varepsilon}} \mathbb{E} \left[\int_{\mathbb{R}^d} |P_h^\varepsilon g_\varepsilon(x + X_t^\varepsilon) - g_\varepsilon(x + X_t^\varepsilon)| dx \right]. \end{aligned}$$

Since v^ε is continuous and using the translational invariance of Lebesgue measure, this finally leads to

$$\|\bar{u}^\varepsilon(t+h, \cdot) - \bar{u}^\varepsilon(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq o_h(1) + e^{\frac{Mt}{\varepsilon}} \|P_h^\varepsilon g_\varepsilon - g_\varepsilon\|_{L^1(\mathbb{R}^d)} \xrightarrow{h \rightarrow 0} 0,$$

which proves that \bar{u}^ε is in $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$.

We now prove that \bar{u}^ε is a mild solution of problem (2.1). First, Markov property gives

$$\begin{aligned} \mathbb{E}_x \left[g_\varepsilon(X_t^\varepsilon) \exp \left(\int_{t-s}^t R(X_\theta^\varepsilon, v_{t-\theta}^\varepsilon) d\theta \right) R(X_{t-s}^\varepsilon, v_s^\varepsilon) \right] \\ = \mathbb{E}_x \left[\mathbb{E}_{X_{t-s}^\varepsilon} \left[g_\varepsilon(X_s^\varepsilon) \exp \left(\int_0^s R(X_\theta^\varepsilon, v_{s-\theta}^\varepsilon) d\theta \right) \right] R(X_{t-s}^\varepsilon, v_s^\varepsilon) \right] \\ = \mathbb{E}_x \left[u(s, X_{t-s}^\varepsilon) R(X_{t-s}^\varepsilon, v_s^\varepsilon) \right]. \end{aligned} \quad (3.3)$$

Now, we have that

$$\bar{u}^\varepsilon(t, x) = \mathbb{E}_x \left[g_\varepsilon(X_t^\varepsilon) \left(\exp \left(\frac{1}{\varepsilon} \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) - 1 \right) \right] + \mathbb{E}_x [g_\varepsilon(X_t^\varepsilon)]$$

and

$$\exp \left(\int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) - 1 = \int_0^t \exp \left(\int_{t-s}^t R(X_u^\varepsilon, v_{t-u}^\varepsilon) du \right) R(X_{t-s}^\varepsilon, v_s^\varepsilon) ds.$$

Hence (3.3) entails

$$\bar{u}^\varepsilon(t, x) = \int_0^t \mathbb{E}_x \left[\bar{u}^\varepsilon(s, X_{t-s}^\varepsilon) R(X_{t-s}^\varepsilon, v_s^\varepsilon) \right] ds + \mathbb{E}_x [g_\varepsilon(X_t^\varepsilon)] = P_t^\varepsilon g_\varepsilon(x) + \int_0^t P_{t-s}^\varepsilon (\bar{u}^\varepsilon(s, x) R(x, v_s^\varepsilon)) ds.$$

This ends the proof that \bar{u}^ε is a mild solution of problem (2.1). \square

Lemma 3.3. *Let $u \in C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ be a mild solution of problem (2.1), then u is a weak solution of problem (2.1).*

Proof. We begin the proof by showing that the mild formulation (3.2) of problem (2.1) admits a unique solution in $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$. This is standard calculations for mild solutions (cf. e.g. [33, 28]) Let u and v be two solutions lying in this space such that $u(0, \cdot) = v(0, \cdot)$. We have

$$\begin{aligned} \|u(t, \cdot) - v(t, \cdot)\|_{L^1(\mathbb{R}^d)} &= \left\| \int_0^t P_{t-s}^\varepsilon [u(s, \cdot) R(\cdot, v_s^\varepsilon) - v(s, \cdot) R(\cdot, v_s^\varepsilon)] ds \right\|_{L^1(\mathbb{R}^d)} \\ &\leq \int_0^t \|P_{t-s}^\varepsilon [u(s, \cdot) R(\cdot, v_s^\varepsilon) - v(s, \cdot) R(\cdot, v_s^\varepsilon)]\|_{L^1(\mathbb{R}^d)} ds \end{aligned}$$

Now, since $(P_t^\varepsilon)_{t \in \mathbb{R}_+}$ is a contractive semi-group in $L^1(\mathbb{R}^d)$, we get, using also Assumption (2d),

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq M \int_0^t \|u(s, \cdot) - v(s, \cdot)\|_{L^1(\mathbb{R}^d)} ds.$$

This proves the assertion through Gronwall lemma.

Now, assume that u is the weak solution of problem (2.1), and consider the following Cauchy problem

$$\begin{cases} \partial_t w = \varepsilon \Delta w + \frac{1}{\varepsilon} u(t, x) R(x, v_t^\varepsilon), \\ w(0, x) = \exp\left(-\frac{h_\varepsilon(x)}{\varepsilon}\right) =: g_\varepsilon(x). \end{cases} \quad (3.4)$$

Since the inhomogeneous term belongs to $L^1(\mathbb{R}^d)$, it is well known that this problem admits a unique weak solution which is given by Duhamel's formula:

$$w(t, x) = P_t^\varepsilon g_\varepsilon(x) + \frac{1}{\varepsilon} \int_0^t P_{t-s}^\varepsilon u(t, x) R(x, v_s^\varepsilon) ds.$$

Since u is the solution of the above problem, we obtain that

$$u(t, x) = P_t^\varepsilon g_\varepsilon(x) + \frac{1}{\varepsilon} \int_0^t P_{t-s}^\varepsilon u(t, x) R(x, v_s^\varepsilon) ds.$$

Hence, the lemma is proved by uniqueness of mild solutions for problem (2.1). \square

4 Small diffusion asymptotic

Recall from Theorem 3.1 that

$$u^\varepsilon(t, x) = \mathbb{E}_x \left[\exp \left(-\varepsilon^{-1} h(X_t^\varepsilon) + \frac{1}{\varepsilon} \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right], \quad (4.1)$$

where $X_t^\varepsilon = x + \sqrt{\varepsilon} B_t$ under \mathbb{P}_x . This formula suggests to apply Varadhan's lemma to study the convergence of $\varepsilon \log u^\varepsilon(t, x)$ as $\varepsilon \rightarrow 0$. Let us fix $t > 0$.

Lemma 4.1. *The function $\Phi_\varepsilon : C([0, t]) \rightarrow \mathbb{R}$ defined by*

$$\Phi_\varepsilon(\varphi) = \int_0^t R(\varphi_s, v_s^\varepsilon) ds$$

is Lipschitz continuous on $C([0, t])$ endowed with the L^∞ -norm, uniformly w.r.t. to ε for ε small enough.

Moreover, there exists a kernel \mathcal{M} on $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^k)$ such that, along a subsequence $(\varepsilon_k)_{k \geq 1}$ converging to 0, we have

$$\Phi(\varphi) := \lim_{k \rightarrow \infty} \Phi_{\varepsilon_k}(\varphi) = \int_0^t \int_{\mathbb{R}^k} R(\varphi_s, y) \mathcal{M}_s(dy) ds, \quad \forall \varphi \in C([0, t])$$

We recall that a kernel \mathcal{M} is a function from $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^d)$ into \mathbb{R}_+ such that, for all $t \in \mathbb{R}_+$, \mathcal{M}_s is a measure on $\mathcal{B}(\mathbb{R}^d)$ and, for all $A \in \mathcal{B}(\mathbb{R}^d)$, the function $s \rightarrow \mathcal{M}_s(A)$ is measurable.

Proof. We begin by showing that $\Phi_\varepsilon(\varphi)$ is continuous for all ε , uniformly w.r.t. ε . Since $R(\cdot, v)$ lies in $W^{2, \infty}$, it follows that it is Lipschitz continuous, uniformly for v is the annulus \mathcal{H} (see Assumption 2d). Hence, for $\psi, \varphi \in C([0, t])$, it follows from Morrey's inequality that

$$\int_0^t |R(\varphi_s, v_s^\varepsilon) - R(\psi_s, v_s^\varepsilon)| ds \leq \int_0^t \|R(\cdot, v_s^\varepsilon)\|_{W^{1, \infty}} \|\varphi_s - \psi_s\| ds \leq t \sup_{s \in [0, t]} \|R(\cdot, v_s^\varepsilon)\|_{W^{1, \infty}} \|\varphi - \psi\|_{L^\infty([0, t])}.$$

Hence, the result follows from (2.3).

Now, fix $T > 0$, and let Γ_T^ε , for all ε , be the measure defined on $([0, T] \times \mathcal{H}, \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathcal{H}))$ by

$$\Gamma_T^\varepsilon(A \times B) = \int_A \mathbb{1}_{v_s^\varepsilon \in B} ds, \quad \forall A \in \mathcal{B}([0, T]), B \in \mathcal{B}(\mathcal{H}).$$

Since $(\Gamma_T^\varepsilon)_{\varepsilon > 0}$ is a family of finite measures defined on a compact metric space, it is weakly precompact. Hence, there exists a subsequence $(\varepsilon_k^T)_{k \geq 1}$ such that, the sequence of measures $(\Gamma_T^{\varepsilon_k^T})_{k \geq 1}$ converges weakly to some measure denoted by Γ_T . It follows from a diagonal argument that there exists a sequence $(\varepsilon_k)_{k \geq 1}$ such that, for any positive integer n , $\Gamma_n^{\varepsilon_k}$ converges weakly to a measure Γ_n . Now, if one wants to define a measure on \mathbb{R}_+ using the family $(\Gamma_n)_{n \geq 1}$, he needs that for $m < n$, the restriction of Γ_n on $[0, m]$ coincides with Γ_m .

To prove this we first remark that, for all $k \in \mathbb{N}$ and $\delta > 0$,

$$\Gamma_n^{\varepsilon_k}((m - \delta, m + \delta) \times \mathcal{H}) = \int_{m - \delta}^{m + \delta} \mathbb{1}_{v_s^{\varepsilon_k} \in \mathcal{H}} ds = 2\delta. \quad (4.2)$$

Let f be a bounded and continuous function on $[0, m] \times \mathcal{H}$ and $(f_\ell)_{\ell \geq 1}$ a sequence of uniformly bounded continuous function on $[0, n] \times \mathcal{H}$ such that

$$f_\ell(t, x) \xrightarrow{\ell \rightarrow \infty} f(t, x) \mathbb{1}_{t \in [0, m]}, \quad \forall (t, x) \in [0, n] \times \mathcal{H},$$

and $f_\ell = f$ on $[0, m - \delta] \times \mathcal{H}$ and $f_\ell = 0$ on $(m + \delta, n] \times \mathcal{H}$. Using (4.2) we have, for all $k, \ell \in \mathbb{N}$,

$$|\Gamma_m^{\varepsilon_k}(f) - \Gamma_n^{\varepsilon_k}(f_\ell)| \leq c\delta,$$

where the constant c does not depend on k and ℓ and $\mu(f)$ is the integral of a function f w.r.t. a measure μ on the corresponding space. Since, in addition, $\Gamma_m^{\varepsilon_k}(f)$ converges to $\Gamma_m(f)$, $\Gamma_n^{\varepsilon_k}(f_\ell)$ converges to $\Gamma_n(f_\ell)$ when k goes to infinity and, by Lebesgue's theorem, $\Gamma_n(f_\ell)$ converges to $\Gamma_n(f)$ when ℓ goes to infinity, we deduce

$$|\Gamma_m(f) - \Gamma_n(f)| \leq c\delta.$$

Since δ was arbitrary we have proved that the restriction of Γ_n on $[0, m]$ coincides with Γ_m . As a consequence, we can define on $\mathbb{R}_+ \times \mathcal{H}$ a measure Γ whose restrictions on $[0, n] \times \mathcal{H}$ is Γ_n for all $n \in \mathbb{N}$.

In particular, for all t , $\Gamma_t^{\varepsilon_k}$ converges weakly to the restriction of Γ to $[0, t] \times \mathcal{H}$. In addition, for all function φ continuous on $[0, t]$, the map $(s, x) \mapsto R(\varphi_s, x)$ is continuous and hence

$$\Phi(\varphi) := \lim_{k \rightarrow \infty} \int_0^t R(\varphi_s, v_s^{\varepsilon_k}) ds = \int_{[0, t] \times \mathbb{R}^d} R(\varphi_s, x) \Gamma(ds, dx), \quad \forall t \leq T.$$

It remains to show that Γ can be disintegrated along $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \lambda)$. Let A be a null set of $[0, T]$, for a fixed positive time T . It follows that there exists, for all $\eta > 0$, a denumerable family of open ball $(B_n^\eta)_{n \geq 1}$ such that

$$A \subset \bigcup_{n=1}^{\infty} B_n^\eta \quad \text{and} \quad \lambda\left(\bigcup_{n=1}^{\infty} B_n^\eta\right) < \eta.$$

Let H be a measurable set of \mathcal{H} , then

$$\Gamma(A \times H) \leq \Gamma_T(\cup_{n \geq 1} B_n^\eta \times \mathcal{H}) \leq \liminf_{k \rightarrow \infty} \Gamma_T^{\varepsilon_k} \left(\bigcup_{n=1}^{\infty} B_n^\eta \times \mathcal{H} \right) < C\eta$$

which implies that $\Gamma(A \times H) = 0$. According to Radon-Nikodym's theorem, there exists, for all H , an integrable function $s \rightarrow \mathcal{M}_s(H)$ such that

$$\Gamma(A \times H) = \int_{\mathbb{R}_+} \mathcal{M}_s(H) ds.$$

Usual theory (cf. e.g. [6]) ensures that there exists a modification of this map such that $H \in \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}_s(H)$ is a measure for almost all $s \in \mathbb{R}_+$. \square

Theorem 4.2. For all (t, x) in $\mathbb{R}_+ \times \mathbb{R}^d$,

$$V(t, x) := \lim_{k \rightarrow \infty} \varepsilon_k \log u^{\varepsilon_k}(t, x) = \sup_{\varphi \in \mathcal{G}_{t,x}} \{-h(\varphi_0) + \Phi(\varphi) - I_t(\varphi)\} \quad (4.3)$$

with

$$I_t(\varphi) = \begin{cases} \int_0^t \|\varphi'(s)\|^2 ds & \text{if } \varphi \text{ is absolutely continuous,} \\ +\infty & \text{otherwise,} \end{cases}$$

$\mathcal{G}_{t,x}$ denotes the set of continuous functions from $[0, t]$ to \mathbb{R}^d such that $\varphi_t = x$, and Φ and $(\varepsilon_k)_{k \geq 1}$ are defined in Lemma 4.1.

Proof. The proof of Theorem 4.2 is a simple adaptation of the classical proof of Varadhan's Lemma (cf. e.g. [13]). The main difference is due to the dependence on ε of the functions h_ε and Φ_ε involved in formula (4.1). We give the details for completeness and for future reference in Section 7. Let μ_ε^x the law of $(X_s^\varepsilon, s \in [0, t])$, when $X_0^\varepsilon = x$ almost surely on $C([0, t])$.

Let $\varphi \in C([0, t])$ such that $\varphi_0 = x$, and $\delta > 0$. Since, Φ_ε and h_ε are both continuous uniformly w.r.t. ε and h_ε converges uniformly to h , there exists a neighborhood G_φ of φ in $C([0, t])$ such that, for ε small enough (independent of φ),

$$\sup_{\psi \in \overline{G}_\varphi} -h_\varepsilon(\psi_t) + \Phi_\varepsilon(\hat{\psi}) - \delta \leq h(\varphi_t) + \Phi_\varepsilon(\hat{\varphi}) \leq \inf_{\psi \in G_\varphi} -h_\varepsilon(\psi_t) + \Phi_\varepsilon(\hat{\psi}) + \delta, \quad (4.4)$$

where we use the notation $\hat{\varphi}_s = \varphi_{t-s}$. In addition, Lemma 4.1 implies that, for k large enough,

$$\Phi(\hat{\varphi}) - \delta \leq \Phi_{\varepsilon_k}(\hat{\varphi}) \leq \Phi(\hat{\varphi}) + \delta.$$

We start with the lower bound: for k large enough,

$$\begin{aligned} u^{\varepsilon_k}(t, x) &\geq \mathbb{E}_x \left[\exp \left(-\frac{h_{\varepsilon_k}(X_t^{\varepsilon_k})}{\varepsilon_k} + \frac{1}{\varepsilon_k} \int_0^t R(X_s^{\varepsilon_k}, v_{t-s}^{\varepsilon_k}) ds \right) \mathbb{1}_{X^{\varepsilon_k} \in G_\varphi} \right] \\ &\geq \mathbb{P}_x(X^{\varepsilon_k} \in G_\varphi) \exp \left(\frac{-h(\varphi_t) + \Phi(\hat{\varphi}) - 3\delta}{\varepsilon_k} \right), \end{aligned}$$

Schilder's theorem (large deviation principle for the Brownian motion [13]) gives

$$\liminf_{k \rightarrow \infty} \varepsilon_k \log(u^{\varepsilon_k}(t, x)) \geq -h(\varphi_t) + \Phi(\hat{\varphi}) - I(\varphi) - 3\delta = -h(\hat{\varphi}_0) + \Phi(\hat{\varphi}) - I(\hat{\varphi}) - 3\delta.$$

Since this holds for all $\delta > 0$, we have proved the lower bound of Varadhan's Lemma along the sequence $(\varepsilon_k)_{k \geq 1}$.

For the upper bound, we use the fact that I is a good rate function, i.e. it is lower semi-continuous with compact level sets. Let a be such that

$$a \leq \sup_{\varphi \in \mathcal{G}_{t,x}} \{-h(\varphi_0) + \Phi(\varphi) - I(\varphi)\}.$$

The set

$$K = \{\varphi \in C([0, t]) : \varphi_0 = x, I(\varphi) \leq Mt + \delta - a\}$$

is compact, where the constant M is defined in Assumption 2d. Hence there exists a finite family of functions $\varphi_1, \dots, \varphi_n$ (for some integer n), such that $\varphi_i(0) = x$ for all $1 \leq i \leq n$ and

$$K \subset \bigcup_{i=1}^n G_{\varphi_i}.$$

Using equation (4.4) and Lemma 4.1, we get

$$\begin{aligned} \limsup_{k \rightarrow \infty} \varepsilon_k \log \mathbb{E}_x \left[\exp \left(-\frac{h_\varepsilon(X_t^{\varepsilon_k})}{\varepsilon_k} + \frac{1}{\varepsilon_k} \int_0^t R(X_s^{\varepsilon_k}, v_{t-s}^{\varepsilon_k}) ds \right) \mathbb{1}_{X^{\varepsilon_k} \in G_{\varphi_i}} \right] \\ \leq h(\varphi_i(t)) + \Phi(\hat{\varphi}_i) - I(\varphi_i) + 2\delta \end{aligned}$$

and, using the fact that $-h_\varepsilon \leq -h + \delta \leq \delta$ for ε small enough,

$$\begin{aligned} \limsup_{k \rightarrow \infty} \varepsilon_k \log \mathbb{E}_x \left[\exp \left(-\frac{h_\varepsilon(X_t^{\varepsilon_k})}{\varepsilon_k} + \frac{1}{\varepsilon_k} \int_0^t R(X_s^{\varepsilon_k}, v_{t-s}^{\varepsilon_k}) ds \right) \mathbb{1}_{X^{\varepsilon_k} \in K^c} \right] \\ \leq Mt + \delta + \limsup_{k \rightarrow \infty} \varepsilon_k \log(\mathbb{P}(X^{\varepsilon_k} \notin K)) \leq a. \end{aligned}$$

Therefore

$$\begin{aligned} \limsup_{k \rightarrow \infty} \varepsilon_k \log(u^{\varepsilon_k}(t, x)) &\leq \max \left\{ a, \sup_{1 \leq i \leq n} h(\hat{\varphi}_i(0)) + \Phi(\hat{\varphi}_i) - I(\hat{\varphi}_i) + 2\delta \right\} \\ &\leq \sup_{\varphi \in \mathcal{G}_{t,x}} \{h(\varphi_0) + \Phi(\varphi) - I(\varphi)\} + 2\delta. \end{aligned}$$

This ends the proof of the upper bound. \square

Thanks to the representation (4.1) of the solution of problem (2.1) the convergence given above can be enhanced.

Theorem 4.3. *The convergence stated in Theorem 4.2 holds uniformly on compact sets and the limit $V(t, x)$ is Lipschitz w.r.t. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.*

This result is an immediate consequence of the following lemma.

Lemma 4.4. *The function*

$$\begin{aligned} \mathbb{R}_+ \times \mathbb{R}^d &\rightarrow \mathbb{R} \\ (t, x) &\mapsto \varepsilon \log u^\varepsilon(t, x), \end{aligned}$$

is Lipschitz w.r.t. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, uniformly w.r.t. ε (at least for ε small enough).

Proof. Let (t, x) and (δ, y) in $\mathbb{R}_+ \times \mathbb{R}^d$. Using (4.1), and writing $X_t^\varepsilon = X_0^\varepsilon + \sqrt{\varepsilon}B_t$ for a Brownian motion B , we get

$$u^\varepsilon(t + \delta, x + y) = \mathbb{E} \left[\exp \left(-\frac{h_\varepsilon(x + y + \sqrt{\varepsilon}B_{t+\delta})}{\varepsilon} + \frac{1}{\varepsilon} \int_0^{t+\delta} R(x + y + \sqrt{\varepsilon}B_s, v_{t+\delta-s}^\varepsilon) ds \right) \right].$$

Now, Markov's property entails

$$\begin{aligned} u^\varepsilon(t + \delta, x + y) &= \mathbb{E} \left[\exp \left(\frac{1}{\varepsilon} \int_0^\delta R(x + y + \sqrt{\varepsilon}B_s, v_{t+\delta-s}^\varepsilon) ds \right) \right. \\ &\quad \left. \times \mathbb{E}_{x+y+\sqrt{\varepsilon}B_\delta} \left[\exp \left(-\frac{h_\varepsilon(X_t^\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon} \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right] \right] \\ &= \mathbb{E} \left[\exp \left(\frac{1}{\varepsilon} \int_0^\delta R(x + y + \sqrt{\varepsilon}B_s, v_{t+\delta-s}^\varepsilon) ds \right) u^\varepsilon(t, x + y + \sqrt{\varepsilon}B_\delta) \right]. \end{aligned}$$

Hence, using Assumption 2d, we get

$$e^{-\frac{M\delta}{\varepsilon}} \mathbb{E} \left[u^\varepsilon(t, x + y + \sqrt{\varepsilon}B_\delta) \right] \leq u^\varepsilon(t + \delta, x + y) \leq e^{\frac{M\delta}{\varepsilon}} \mathbb{E} \left[u^\varepsilon(t, x + y + \sqrt{\varepsilon}B_\delta) \right]. \quad (4.5)$$

In addition, taking into account that $x \rightarrow R(x, v)$ is Lipschitz uniformly w.r.t. $v \in \mathcal{H}$ with Lipschitz norm bounded by M , we have

$$\begin{aligned} u^\varepsilon(t, x + y) &= \mathbb{E} \left[\exp \left(-\frac{h_\varepsilon(x + y + \sqrt{\varepsilon}B_t)}{\varepsilon} + \frac{1}{\varepsilon} \int_0^t R(x + y + \sqrt{\varepsilon}B_s, v_{t-s}^\varepsilon) ds \right) \right] \\ &\leq u^\varepsilon(t, x) \mathbb{E} \left[\exp \left(-\frac{h_\varepsilon(x + y + \sqrt{\varepsilon}B_t) - h_\varepsilon(x + \sqrt{\varepsilon}B_t)}{\varepsilon} + \frac{tM\|y\|}{\varepsilon} \right) \right], \end{aligned}$$

and

$$u^\varepsilon(t, x + y) \geq u^\varepsilon(t, x) \mathbb{E} \left[\exp \left(-\frac{h_\varepsilon(x + y + \sqrt{\varepsilon}B_t) - h_\varepsilon(x + \sqrt{\varepsilon}B_t)}{\varepsilon} - \frac{tM\|y\|}{\varepsilon} \right) \right].$$

Since h_ε is Lipschitz uniformly w.r.t. $\varepsilon > 0$,

$$u^\varepsilon(t, x + y) \leq u^\varepsilon(t, x) e^{\frac{tM + \sup_\varepsilon \|h_\varepsilon\|_{\text{Lip}}}{\varepsilon} \|y\|}.$$

Now, using (4.5), we get

$$u^\varepsilon(t + \delta, x + y) \leq u^\varepsilon(t, x) e^{\frac{M\delta}{\varepsilon}} \mathbb{E} \left[e^{\frac{tM + \sup_\varepsilon \|h_\varepsilon\|_{\text{Lip}}}{\varepsilon} \|y + \sqrt{\varepsilon}B_\delta\|} \right].$$

Then, using the inequality $\|y + \sqrt{\varepsilon}B_\delta\| \leq \|y\| + C\sqrt{\varepsilon}\sum_i |B_\delta^i|$ and elementary computations, we obtain

$$u^\varepsilon(t + \delta, x + y) \leq u^\varepsilon(t, x) e^{\frac{M\delta}{\varepsilon}} e^{\frac{tM + \sup_\varepsilon \|h_\varepsilon\|_{\text{Lip}}}{\varepsilon} (\|y\| + nC\delta)}.$$

Hence,

$$\varepsilon \log u^\varepsilon(t + \delta, x + y) - \varepsilon \log u^\varepsilon(t, x) \leq M\delta + \left(tM + \sup_\varepsilon \|h_\varepsilon\|_{\text{Lip}} \right) (\|y\| + nC\delta).$$

The lower bound is obtained similarly and ends the proof. \square

In the last results, one would like to be able to characterize the limit measure \mathcal{M}_s in terms of the zeroes of $V(s, \cdot)$, in order to obtain a closed form of the optimization problem. Results on this question are known in models with a single resource ($r = 1$) [35, 3, 27], but the known results when $r \geq 2$ [8, 7] require stringent assumptions on the structure of the model, and indeed, it is possible to construct examples where there exist several measures \mathcal{M}_s satisfying the metastability condition of [8]. Since our assumptions are more general, we cannot expect to obtain such results in full generality, so we will focus in Sections 7 and 8 on a case where precise results can be obtained, granting uniqueness for the optimization problem (4.3).

5 Extensions of Theorem 4.2

The proof of Theorem 4.2 only makes use of few properties of the Brownian motion and of the Laplace operator used to model mutations in (2.1). In particular, one expects that it may hold true for partial differential equations of the form

$$\partial_t u^\varepsilon(t, x) = L_\varepsilon u^\varepsilon(t, x) + \frac{1}{\varepsilon} u^\varepsilon(t, x) R(x, v_t^\varepsilon), \quad \forall t > 0, x \in \mathbb{R}^d, \quad (5.1)$$

where L_ε is a linear operator describing mutations.

For our approach to work in this situation, the probabilistic interpretation of Theorem 3.1 must extend to this case, and Varadhan's lemma must be applied as in the proof of Theorem 4.2. For this, one needs that

1. the operator L_ε is the infinitesimal generator of a Markov process $(X_t^\varepsilon, t \geq 0)$;
2. existence and uniqueness of a weak solution hold for the partial differential equation (5.1) in an appropriate functional space, for example in $C^1(\mathbb{R}_+, L^1(\mathbb{R}^d))$ as in Theorem 2.2, and any $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ mild solution to the PDE must be a weak solution;
3. the method used to prove the Feynman-Kac formula of Theorem 3.1 applies, in other words the function

$$\bar{u}^\varepsilon(t, x) = \mathbb{E}_x \left[\exp \left(-\frac{h_\varepsilon(X_t^\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon} \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right] \quad (5.2)$$

can be shown to be $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ (note that the proof that \bar{u} is a mild solution to the PDE only relies on the Markov property, so it is true in general);

4. the family of Markov processes $(X^\varepsilon)_{\varepsilon>0}$ must satisfy a large deviations principle with rate ε^{-1} and a good rate function, i.e. a lower semicontinuous function with compact level sets.

Note that the compactness argument of Lemma 4.1 does not depend on the mutation operator. It only follows from our assumptions on R .

For example, all these points apply to the problem

$$\partial_t u^\varepsilon(t, x) = \frac{1}{\varepsilon} \int_{\mathbb{R}^n} [u^\varepsilon(t, x + \varepsilon z) - u^\varepsilon(t, x)] K(z) dz + \frac{1}{\varepsilon} u^\varepsilon(t, x) R(x, v_t^\varepsilon), \quad (5.3)$$

where $K : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is such that

$$\int_{\mathbb{R}^n} z K(z) dz = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} e^{\langle a, z \rangle^2} K(z) dz < \infty \quad \text{for all } a \in \mathbb{R}^n.$$

This form of mutation operator has already been studied in [8, 4, 3]. Similar equations are also considered in [24, 37, 14]. In this case, we can check all the points above as follows:

1. The Markov process X^ε is a continuous-time random walk, with jump rate $\frac{\|K\|_{L^1}}{\varepsilon}$ and i.i.d. jump steps distributed as εZ , where the random variable Z has law $\frac{K(z)}{\|K\|_{L^1}} dz$.
2. Existence and uniqueness of the solution to (5.3) in $C^1(\mathbb{R}_+, L^1(\mathbb{R}^d))$ follows from [3] (this is the point where the finiteness of quadratic exponential moments of K is needed).
3. Since random walks are shift-invariant as Brownian motion, the regularity of (5.2) can be proved exactly as in the proof of Theorem 3.1.
4. It is well-known (see [18, Section 10.3]) that, under the previous assumptions, the family $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviation principle with rate ε^{-1} and good rate function on $\mathbb{D}([0, t], \mathbb{R}^n)$, the set of càdlàg functions from $[0, t]$ to \mathbb{R}^n , given by

$$I_t(\varphi) = \begin{cases} \int_0^t \int_{\mathbb{R}^d} (e^{\langle z, \dot{\varphi}_s \rangle} - 1) K(z) dz ds & \text{if } \varphi \text{ is absolutely continuous,} \\ +\infty & \text{otherwise.} \end{cases}$$

We will give in Section 7 another example of extension, in the case where the Markov processes X^ε take values in a finite set. However, this case will raise specific difficulties because the rate function of the associated large deviations principle has non-compact level sets.

6 Links between the variational and Hamilton-Jacobi problems

As explained in the introduction, the convergence of $\varepsilon \log u^\varepsilon$ has been studied in various works using PDE approaches, with a limit solving a Hamilton-Jacobi problem with constraints. With our approach, instead of a Hamilton-Jacobi equation, we obtain a variational characterization of the limit, under assumptions on R slightly weaker than those of [35, 3, 27] and valid for any value of the parameter r , biologically interpreted as a number of resources (see the introduction). Therefore, we obtain naturally the identification between a solution to Hamilton-Jacobi problems and a variational problem. For example, the next result is a consequence of [27, Eq. (5.8)].

Corollary 6.1. *In addition to the assumptions of Section 2, assume that $r = 1$, h_ε is C^2 uniformly in $\varepsilon > 0$ and there exist constants A, B, C, D such that, for all $x \in \mathbb{R}^d$, $v_{\min} \leq \|v\|_1 \leq v_{\max}$ and $\varepsilon > 0$,*

$$\begin{aligned} -A|x|^2 &\leq R(x, v) \leq B - A^{-1}|x|^2, \\ -B + A^{-1}|x|^2 &\leq h_\varepsilon(x) \leq B + A|x|^2, \\ -C\text{Id} &\leq D^2R(x, v) \leq -C^{-1}\text{Id}, \\ C^{-1}\text{Id} &\leq D^2h_\varepsilon(x) \leq C\text{Id}, \\ \Delta(\Psi_1R) &\geq -D, \end{aligned}$$

where the third and fourth inequalities have to be understood in the sense of symmetric matrices and Id is the n -dimensional identity matrix. Then, v^{ε_k} converges in $L^1_{\text{loc}}(\mathbb{R}_+)$ to a nondecreasing limit \bar{v} along the subsequence ε_k of Lemma 4.1, the kernel \mathcal{M} satisfies

$$\mathcal{M}_s(dy) = \delta_{\bar{v}(s)}(dy), \quad \forall s \geq 0, \quad (6.1)$$

and the limit V of Theorem 4.2 solves in the viscosity sense

$$\begin{cases} \partial_t V(t, x) = R(x, \bar{v}_t) + \frac{1}{2}|\nabla V(t, x)|, & \forall t \geq 0, x \in \mathbb{R}^d, \\ \max_{x \in \mathbb{R}^d} V(t, x) = 0, & \forall t \geq 0. \end{cases} \quad (6.2)$$

The only point of this corollary that is not a direct consequence of [27] is the convergence of the full sequence $(v^{\varepsilon_k})_{k \geq 1}$ to \bar{v} . We know from [27] that such a convergence is true for a subsequence and hence (6.1) holds true. Since, again from [27], any subsequence of v^{ε_k} admits a subsequence converging in L^1 and any accumulation point in L^1 must be \bar{v} by (6.1), the result is clear.

The last corollary provides a variational interpretation (4.3) for the solution to (6.2) given by the limit of $\varepsilon_k \log u^{\varepsilon_k}$. Since uniqueness for (4.3) is not known in general (note however that it is known under additional assumptions, see [32]), we cannot deduce that any solution to (6.2) admits a variational formulation.

Note that Hamilton-Jacobi equations are known to be related to variational problems appearing in control theory. However, this link is known in general in cases with continuous and time-independent coefficients in the Hamilton-Jacobi equation [26, 19]. In our case, it is known that \bar{v} may be discontinuous in general, hence Corollary 6.1 cannot be deduced from these general results. Hamilton-Jacobi problems with discontinuous coefficients were studied in several works but none covers the form of the optimization problem given by our results: first, the question of appropriate notion of viscosity solutions of irregular Hamilton-Jacobi problems has been studied by several authors starting from [25, 1]. In the later reference, these notions are only used to study variational problems without integral term and with continuous Hamiltonian without time-dependencies. In the book [11], problems of calculus of variation with irregular Lagrangian are studied, providing existence of optimal solutions but no link with the Hamilton-Jacobi problem. Finally, in the article [38], optimal control problems with measurable in time Hamiltonian are studied but without the integral term in the optimisation problem as in (4.3). Hence Corollary 6.1 seems to give, through an indirect approach, an original optimization formulation for Hamilton-Jacobi problems with constraints as those of [16, 27].

Other mutation operators were also studied with the PDE approach. Up to our knowledge, the only work providing the Hamilton-Jacobi limit with several resources is [8]. In this case, we obtain the following corollary.

Corollary 6.2. *Assume that $d = 1$, $r \geq 1$ is arbitrary and R is given by Equation (1.2) with $d(x) \equiv 1$. Consider the solution u^ε of the PDE (5.3) with this function R and with a function K in C_c^∞ and such that $\int_{\mathbb{R}} zK(z) dz = 0$. Assume that the assumptions of Section 2 are satisfied, that Ψ_i is C_b^2 for all $1 \leq i \leq r$, that $\sum_{i=1}^r c_i \Psi_i(x) \geq 1$ for all $x \in \mathbb{R}$ and*

$$\exists \bar{r} \leq r, \forall v_1 \in [0, c_1], \dots, v_r \in [0, c_r], \text{ the function } x \mapsto \sum_{i=1}^r v_i \Psi_i(x) - 1 \text{ has at most } \bar{r} \text{ roots,}$$

$$\forall x_1, \dots, x_{\bar{r}} \text{ distinct, the } \bar{r} \text{ vectors } \Psi(x_1), \dots, \Psi(x_{\bar{r}}) \text{ are linearly independent,}$$

where $\Psi(x) = (\Psi_1(x), \dots, \Psi_r(x))$. Assume also that h_ε is C^2 for all $\varepsilon > 0$, $h_\varepsilon - h$ converges to 0 in $W^{1,\infty}(\mathbb{R})$,

$$\sup_{\varepsilon > 0} \|\partial_x h_\varepsilon\|_{L^\infty(\mathbb{R})} < \infty \quad \text{and} \quad \inf_{\varepsilon > 0} \inf_{x \in \mathbb{R}} \partial_{xx} h_\varepsilon(x) > -\infty.$$

Then, v^{ε_k} converges in L_{loc}^1 to \bar{v} along the subsequence ε_k of Lemma 4.1, where $\bar{v}(t) = F(\{V(t, \cdot) = 0\})$ for an explicit function F (see [8, Prop. 1.1]), the kernel \mathcal{M} satisfies

$$\mathcal{M}_s(dy) = \delta_{\bar{v}(s)}(dy), \quad \forall s \geq 0,$$

and the limit V of Theorem 4.2 is such that $\varphi(t, x) = V(t, x) - \sum_{i=1}^r \int_0^t \bar{v}_i(s) ds \Psi_i(x)$ solves in the viscosity sense

$$\begin{cases} \partial_t \varphi(t, x) = H\left(\partial_x \varphi(t, x) + \sum_{i=1}^r \int_0^t \bar{v}_i(s) ds \Psi_i(x)\right), & \forall t \geq 0, x \in \mathbb{R}^d, \\ \max_{x \in \mathbb{R}^d} \left\{ \varphi(t, x) + \sum_{i=1}^r \int_0^t \bar{v}_i(s) ds \Psi_i(x) \right\} = 0, & \forall t \geq 0, \end{cases}$$

where

$$H(p) = \int_{\mathbb{R}} (e^{pz} - 1) K(z) dz.$$

7 The finite case

We consider here another extension of Theorem 4.2 to processes satisfying a large deviation principle, in the case where the trait space is finite. Since in this case the equation satisfied by the limit V is simpler, we are able to study it in details in Section 8. We consider a finite set E and the system of ordinary differential equations

$$\begin{cases} \dot{u}^\varepsilon(t, i) = \sum_{j \in E \setminus \{i\}} \exp\left(-\frac{\mathfrak{I}(i, j)}{\varepsilon}\right) (u^\varepsilon(t, j) - u^\varepsilon(t, i)) + \frac{1}{\varepsilon} u^\varepsilon(t, i) R(i, v_i^\varepsilon), & \forall t \in [0, T], \forall i \in E \\ u^\varepsilon(0, i) = \exp\left(-\frac{h^\varepsilon(i)}{\varepsilon}\right), \end{cases} \quad (7.1)$$

where $\mathfrak{T}(i, j) \in (0, +\infty]$ for all $i \neq j \in E$, $R(i, u) : E \times \mathbb{R}^r \mapsto \mathbb{R}$, $h^\varepsilon : E \rightarrow \mathbb{R}$ and $v_t^\varepsilon = (v_t^{1,\varepsilon}, \dots, v_t^{r,\varepsilon})$ is defined by

$$v_t^{i,\varepsilon} = \sum_{j \in E} u(t, j) \Psi_i(j), \quad \forall 1 \leq i \leq r,$$

for some functions $\Psi_i : E \mapsto (0, +\infty)$. The term $e^{-\varepsilon^{-1}\mathfrak{T}(i,j)}$ corresponds to the mutation rate from trait i to trait j . Its value is by convention 0 when $\mathfrak{T}(i, j) = +\infty$, which means that mutations are impossible from state i to state j .

The standing assumptions on Ψ , R and h_ε given in Section 2 are still assumed to hold true here replacing \mathbb{R}^d by E (except of course for the assumptions of regularity in the trait space). We also need the following assumption.

Assumption on \mathfrak{T}

For all distinct $i, j, k \in E$,

$$\mathfrak{T}(i, j) + \mathfrak{T}(j, k) > \mathfrak{T}(i, k) \tag{7.2}$$

(with the convention $\infty > \infty$) and we set

$$\eta := \inf_{i,j,k \in E \text{ distinct s.t. } \mathfrak{T}(i,k) < \infty} \mathfrak{T}(i, j) + \mathfrak{T}(j, k) - \mathfrak{T}(i, k) > 0. \tag{7.3}$$

Similarly as for problem (1.1), the solution of problem (7.1) remains bounded. This is given in the following Lemma.

Lemma 7.1. *We have for all $t \geq 0$*

$$\frac{v_{\min} - A^{-1}(|E| - 1)e^{-\beta/\varepsilon}}{\Psi_{\max}} \leq \sum_{i \in E} u^\varepsilon(t, i) \leq \frac{v_{\max} + A(|E| - 1)e^{-\gamma/\varepsilon}}{\Psi_{\min}}$$

holds true for any positive t as far as it holds for $t = 0$, where $|E|$ stands for the cardinality of E ,

$$\gamma = \inf\{\mathfrak{T}(i, j) \mid i, j \in E, i \neq j\} > 0, \quad \beta = \sup\{\mathfrak{T}(i, j) \mid i, j \in E, i \neq j\}$$

and the constants A , v_{\min} , v_{\max} , Ψ_{\min} and Ψ_{\max} are defined in Section 2.

Hence we shall also assume in the sequel that

$$\frac{v_{\min} - A^{-1}(|E| - 1)e^{-\beta/\varepsilon}}{\Psi_{\max}} \leq \sum_{i \in E} e^{-h_\varepsilon(i)/\varepsilon} \leq \frac{v_{\max} + A(|E| - 1)e^{-\gamma/\varepsilon}}{\Psi_{\min}}.$$

We also define the compact set

$$\mathcal{H} := \left\{ u \in \mathbb{R}_+^E : \frac{v_{\min} - A^{-1}(|E| - 1)e^{-\beta/\varepsilon}}{\Psi_{\max}} \leq \|u\|_1 \leq \frac{v_{\max} + A(|E| - 1)e^{-\gamma/\varepsilon}}{\Psi_{\min}} \right\}$$

which is invariant for the dynamics (7.1).

Proof. Set, for any positive real number t ,

$$u^\varepsilon(t) = \sum_{i \in E} u^\varepsilon(t, i).$$

According to (7.1), we get

$$\dot{u}^\varepsilon(t) \leq u^\varepsilon(t) \max_{i \in E} R(i, v_t^\varepsilon) + \sum_{i \in E} \sum_{j \in E \setminus \{i\}} \exp\left(-\frac{\mathfrak{T}(i, j)}{\varepsilon}\right) (u^\varepsilon(t, j) - u^\varepsilon(t, i)).$$

Hence

$$u^\varepsilon(t) \leq u^\varepsilon(t) \left(\max_{i \in E} R(i, v_t^\varepsilon) + (|E| - 1)e^{-\frac{\gamma}{\varepsilon}} \right).$$

Now, using Hypothesis (2b) and (2c), we obtain that

$$\max_{i \in E} R(i, v_t^\varepsilon) < -(|E| - 1)e^{-\frac{\gamma}{\varepsilon}}$$

as soon as $v_t^{i, \varepsilon} > v_{\max} + A(|E| - 1)e^{-\frac{\gamma}{\varepsilon}}$, for some i in $\{1, \dots, r\}$. Using Hypothesis (2.2), we deduce that

$$u^\varepsilon(t) \leq \frac{1}{\Psi_{min}} \left(v_{\max} + A(|E| - 1)e^{-\frac{\gamma}{\varepsilon}} \right)$$

as soon as this holds at time $t = 0$. The proof of the lower bound is similar. \square

Our first goal is to describe the solution u^ε of the system using an integral representation similar to (4.1). Let $(X_s^\varepsilon, s \in [0, T])$ be the Markov processes in E with infinitesimal generator

$$L^\varepsilon f(i) = \sum_{j \in E} (f(j) - f(i)) e^{-\frac{\mathfrak{T}(i, j)}{\varepsilon}},$$

i.e. the Markov process which jumps from state $i \in E$ to $j \neq i$ with rate $\exp(-\mathfrak{T}(i, j)/\varepsilon)$.

Proposition 7.2. (*Integral representation*) For any positive real number t and any element i of E , we have

$$u^\varepsilon(t, i) = \mathbb{E}_i \left[\exp \left(-\frac{h^\varepsilon(X_t^\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon} \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right].$$

Proof. First note that the part of the proof of Lemma 3.2 showing that (4.1) is a mild solution of problem (2.1) do not rely in any manner on the Brownian nature of B^ε . As a consequence, one can directly deduce that $u^\varepsilon(t, i)$ satisfies

$$u^\varepsilon(t, i) = P_t^\varepsilon(e^{-h^\varepsilon/\varepsilon})(i) + \int_0^t P_{t-s}^\varepsilon(u^\varepsilon(s, i)R(i, v_s^\varepsilon)) ds,$$

where $(P_t^\varepsilon, t \in \mathbb{R}_+)$ now stands for the semigroup generated by L^ε (i.e. a simple exponential of matrix since E is finite). This last expression is the Duhamel formulation of (7.1). \square

The next result proves a weak large deviations principle (i.e. a large deviations principle with upper bounds only for compact sets of $\mathbb{D}([0, T], E)$) for the laws of X^ε .

Proposition 7.3. (Weak LDP) $(X^\varepsilon)_{\varepsilon \geq 0}$ satisfies a weak LDP with rate function

$$\begin{aligned} I_T : \mathbb{D}([0, T], E) &\mapsto \mathbb{R} \\ \varphi &\rightarrow \sum_{l=1}^{N_\varphi} \mathfrak{I}(\varphi_{t_l^\varphi-}, \varphi_{t_l^\varphi}), \end{aligned}$$

where $\mathbb{D}([0, T], E)$ is the space of càdlàg functions from $[0, T]$ to E and N_φ is the number of jumps of φ and $(t_l^\varphi)_{1 \leq l \leq N_\varphi}$ the increasing sequence of jump times of φ .

We shall also make use of the notation

$$I_T(\varphi) = \sum_{0 < s \leq T} \mathfrak{I}(\varphi_{s-}, \varphi_s)$$

with the implicit convention that $\mathfrak{I}(i, i) = 0$ for all $i \in E$.

Before proving this result, we focus on the following lemma which provides a convenient topological basis of the space $\mathbb{D}([0, T], E)$ equipped with the Skorohod topology.

Lemma 7.4. For all $\varphi \in \mathbb{D}([0, T], E)$ and $\delta < 1$ define

$$B_{Sko}(\varphi, \delta) = \{\psi \in \mathbb{D}([0, T], E) \mid N_\varphi = N_\psi, |t_i^\varphi - t_i^\psi| < \delta, \varphi_0 = \psi_0, \text{ and } \varphi_{t_i^\varphi} = \psi_{t_i^\psi}\}$$

Then, the set

$$\{B_{Sko}(\varphi, \varepsilon) \mid \varepsilon \in [0, 1], \varphi \in \mathbb{D}([0, T], E)\}$$

is a topological basis of $\mathbb{D}([0, T], E)$.

Proof. To prove the result it is enough to show that, for all $\delta < 1$ and $\varphi \in \mathbb{D}([0, T], E)$ the set $B_{Sko}(\varphi, \delta)$ is exactly the δ neighborhood of φ for a particular metric inducing the Skorokhod topology. We recall (see e.g. [5]) that the Skorokhod topology can be defined through the metric d_S given by

$$d_S(\varphi, \psi) = \inf_{\lambda \in \Lambda} \left\{ \max \left(\|\lambda - I\|_{L^\infty([0, T])}, \sup_{t \in [0, T]} d(\varphi_t, \psi \circ \lambda_t) \right) \right\}$$

where Λ is the set of continuous increasing functions on $[0, T]$ with $\lambda_0 = 0$ and $\lambda_T = T$ and the distance d on E is defined as $d(i, j) = \mathbb{1}_{i \neq j}$. Let ψ and φ be such that $d_S(\varphi, \psi) < \delta$ for some $\delta < 1$, then there exists λ in Λ such that

$$\begin{cases} |\lambda_s - s| < \delta, & \forall s \in [0, T], \\ d(\varphi_s, \psi(\lambda_s)) < \delta, & \forall s \in [0, T]. \end{cases}$$

Since $\inf_{i, j \in E, i \neq j} d(i, j) = 1$, we have

$$\varphi_t = \psi(\lambda_t), \quad \forall t \in [0, T].$$

Hence, it follows that N_φ equals N_ψ and $t_i^\varphi = \lambda(t_i^\psi)$, for all i . Consequently, $|t_i^\varphi - t_i^\psi| < \delta$, for all i , and $\varphi_{t_i^\varphi} = \psi_{t_i^\psi}$. \square

We can now prove the weak large deviations principle for X^ε .

Proof of Proposition 7.3. Let $(S_i)_{i \geq 0}$ be the discrete time Markov chain associated to X^ε and let T_i be the i -th inter-jump time, i.e. $T_i = J_{i+1} - J_i$ for all $i \geq 0$, where J_i is the i -th jump time of X^ε , for $i \geq 1$ and $J_0 = 0$. We recall that the Markov chain $(T_i, S_i)_{i \geq 0}$ has transition kernel given by

$$P((t, i), ds, dj) = \exp\left(-\frac{\mathfrak{I}(i, j)}{\varepsilon}\right) \exp(-c^\varepsilon(i)s) ds \otimes C(dj), \quad \forall (t, i) \in [0, T] \times E,$$

where

$$c^\varepsilon(i) = \sum_{j \in E, j \neq i} \exp\left(-\frac{\mathfrak{I}(i, j)}{\varepsilon}\right)$$

and $C(dj)$ is the counting measure on E and initial value $(T_0, S_0) = (0, i)$ \mathbb{P}_i -almost surely.

Let φ be an element of $\mathbb{D}([0, T], E)$ such that $\varphi(0) = i$ and $\delta < 1$. We set $t_0^\varphi = 0$. According to Lemma 7.4,

$$\begin{aligned} & \mathbb{P}_i(X^\varepsilon \in B_{\text{Sko}}(\varphi, \delta)) \\ & \leq \mathbb{P}_i \left(\bigcap_{\ell=0}^{N_\varphi-1} \left\{ T_\ell \in [t_{\ell+1}^\varphi - t_\ell^\varphi - 2\delta, t_{\ell+1}^\varphi - t_\ell^\varphi + 2\delta], S_\ell = \varphi_{t_\ell^\varphi} \right\} \cap \left\{ T_{N_\varphi} \geq T - t_{N_\varphi} - \delta, S_{N_\varphi} = \varphi_{t_{N_\varphi}^\varphi} \right\} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{P}_i(X^\varepsilon \in B_{\text{Sko}}(\varphi, \delta)) & \leq \left(\prod_{\ell=0}^{N_\varphi-1} \int_{(t_{\ell+1}^\varphi - t_\ell^\varphi - 2\delta) \vee 0}^{t_{\ell+1}^\varphi - t_\ell^\varphi + 2\delta} \exp\left(-\frac{\mathfrak{I}(\varphi_{t_\ell^\varphi}, \varphi_{t_{\ell+1}^\varphi})}{\varepsilon}\right) \exp(-c^\varepsilon(\varphi_{t_\ell^\varphi})s_\ell) ds_\ell \right) \\ & \quad \times \int_{(T - t_{N_\varphi}^\varphi - \delta) \vee 0}^{\infty} c^\varepsilon(\varphi_{t_{N_\varphi}^\varphi}) \exp(-c^\varepsilon(\varphi_{t_{N_\varphi}^\varphi})s) ds \\ & = \prod_{\ell=0}^{N_\varphi-1} \exp\left(-\frac{\mathfrak{I}(\varphi_{t_\ell^\varphi}, \varphi_{t_{\ell+1}^\varphi})}{\varepsilon}\right) e^{-c^\varepsilon(\varphi_{t_\ell^\varphi})(t_{\ell+1}^\varphi - t_\ell^\varphi - 2\delta) \vee 0} \frac{\left(1 - e^{-4\delta c^\varepsilon(\varphi_{t_\ell^\varphi})}\right)}{c^\varepsilon(\varphi_{t_\ell^\varphi})} \\ & \quad \times e^{-c^\varepsilon(\varphi_{t_{N_\varphi}^\varphi})(T - t_{N_\varphi}^\varphi - \delta) \vee 0}. \end{aligned}$$

Now, using the facts that $c^\varepsilon(i) \leq e^{-\frac{c}{\varepsilon}}$ with $c > 0$ and $(1 - \exp(-4\delta c^\varepsilon(\varphi_{t_\ell^\varphi}))) / c^\varepsilon(\varphi_{t_\ell^\varphi}) \rightarrow 4\delta$ when $\varepsilon \rightarrow 0$, we get

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_i(X^\varepsilon \in B_{\text{Sko}}(\varphi, \delta)) \leq - \sum_{\ell=0}^{N_\varphi-1} \mathfrak{I}(\varphi_{t_\ell^\varphi}, \varphi_{t_{\ell+1}^\varphi}).$$

Similarly, for $\delta > 0$ small enough, using the bound

$$\mathbb{P}_i(X^\varepsilon \in B_{\text{Sko}}(\varphi, \delta)) \geq \mathbb{P}_i \left(\bigcap_{\ell=0}^{N_\varphi} \left\{ T_\ell \in \left(t_{\ell+1}^\varphi - t_\ell^\varphi - \frac{\delta}{N_\varphi}, t_{\ell+1}^\varphi - t_\ell^\varphi \right), S_\ell = \varphi_{t_\ell^\varphi} \right\} \right),$$

we obtain

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_i(X^\varepsilon \in B_{\text{Sko}}(\varphi, \delta)) \geq - \sum_{\ell=0}^{N_\varphi-1} \mathfrak{I}(\varphi_{t_\ell^\varphi}, \varphi_{t_{\ell+1}^\varphi}).$$

Hence

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_i(X^\varepsilon \in B_{\text{Sko}}(\varphi, \delta)) = - \sum_{\ell=0}^{N_\varphi-1} \mathfrak{I}(\varphi_{t_\ell^\varphi}, \varphi_{t_{\ell+1}^\varphi}).$$

This classically entails (see e.g. [13, Thm. 4.1.11]) that X^ε satisfies a weak large deviations principle with rate function I_t . \square

Usually, a full large deviations principle is deduced from a weak one using exponential tightness of the laws of X^ε . However, in our case, exponential tightness does not hold. This is due to the fact that the function I_T is not a good rate function, as can be seen from the following example: let i and j be two elements of E and s a real number in $(0, T)$. Now, define for any positive integer n large enough,

$$\varphi_n(u) = \begin{cases} i & \text{if } u \in [0, s) \\ j & \text{if } u \in [s, s + \frac{1}{n}) \\ i & \text{if } u \in [s + \frac{1}{n}, T]. \end{cases}$$

Then, the subset $\{\varphi_n \mid n \in \mathbb{N} \setminus \{0\}\}$ is clearly non compact in $\mathbb{D}([0, T], E)$ while I_T is bounded on this set.

To prove the full large deviations principle, we need the following lemma.

Lemma 7.5. *For all $N \geq 1$ and $t > 0$, we denote by N_t^ε the number of jumps of X^ε before t . There exists a constant $C_{N,t} \geq 0$ such that, for all $i \in E$,*

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_i(N_t^\varepsilon \geq N) \leq -C_{N,t},$$

and for all $t > 0$,

$$\lim_{N \rightarrow \infty} C_{N,t} = +\infty.$$

Proof. Let us fix $x_0, x_1, \dots, x_N \in E$ such that $\mathfrak{I}(x_i, x_{i+1}) > 0$ for all i . We compute

$$\begin{aligned} & \mathbb{P}_{x_0}(N_t^\varepsilon \geq N, X_{J_i}^\varepsilon = x_i, \forall i \in \{0, \dots, N\}) \\ &= \int_0^t \exp\left(-\frac{\mathfrak{I}(x_0, x_1)}{\varepsilon}\right) \exp(-c^\varepsilon(x_0)s_0) ds_0 \dots \\ & \quad \dots \int_0^{t-s_0-\dots-s_{N-2}} \exp\left(-\frac{\mathfrak{I}(x_{N-1}, x_N)}{\varepsilon}\right) \exp(-c^\varepsilon(x_{N-1})s_{N-1}) ds_{N-1} \\ &\leq \exp\left(-\frac{\mathfrak{I}(x_0, x_1) + \dots + \mathfrak{I}(x_{N-1}, x_N)}{\varepsilon}\right) \int_0^t ds_0 \dots \int_0^{t-s_0-\dots-s_{N-2}} ds_{N-1} \\ &= \frac{t^N}{N!} \exp\left(-\frac{\mathfrak{I}(x_0, x_1) + \dots + \mathfrak{I}(x_{N-1}, x_N)}{\varepsilon}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_{x_0}(N_t^\varepsilon \geq N, X_{J_i}^\varepsilon = x_i, \forall i \in \{0, \dots, N\}) \\ \leq -\mathfrak{I}(x_0, x_1) - \dots - \mathfrak{I}(x_{N-1}, x_N) \leq -N \inf_{i, j \in E} \mathfrak{I}(i, j) \end{aligned}$$

Since the number of choices of $x_0, x_1, \dots, x_N \in E$ is finite, we have proved Lemma 7.5. \square

We can now prove that X^ε satisfies (a strong version of) the full LDP.

Theorem 7.6. *For any measurable set F of $\mathbb{D}([0, t], E)$,*

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_i(X^\varepsilon \in F) \leq - \inf_{\varphi \in F} I_t(\varphi) \leq - \inf_{\varphi \in \bar{F}} I_t(\varphi).$$

In particular, X^ε satisfies the large deviation principle with rate ε^{-1} and rate function I_t .

Proof. Let $F \subset \mathbb{D}([0, t], E)$ be measurable. Set, for any positive integer n ,

$$F_n^+ = \{\varphi \in F \mid N_\varphi \geq n\} \text{ and } F_n^- = F \setminus F_n^+.$$

We have

$$\mathbb{P}_i(X^\varepsilon \in F_n^-) = \sum_{\ell=0}^{n-1} \sum_{(x_1, \dots, x_\ell) \in E^\ell} \mathbb{P}_i(\{X_{J_k}^\varepsilon = x_k, \forall k \leq N_t^\varepsilon, N_t^\varepsilon = \ell\} \cap F).$$

According to the computations made in the proof of Lemma 7.5, we have

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_i(X_{J_k}^\varepsilon = x_k, \forall k \leq N_t^\varepsilon, N_t^\varepsilon = \ell) \leq - \left(\mathfrak{I}(i, x_1) + \sum_{k=1}^{\ell-1} \mathfrak{I}(x_k, x_{k+1}) \right),$$

which leads to

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_i(X^\varepsilon \in F_n^-) \leq \max \left\{ - \left(\mathfrak{I}(i, x_1) + \sum_{k=1}^{\ell-1} \mathfrak{I}(x_k, x_{k+1}) \right) \right\},$$

where the maximum is taken with respect to $\ell \in \{0, \dots, n-1\}$ and to the elements (x_1, \dots, x_ℓ) of $\cup_{\ell=0}^n E^\ell$ such that

$$\{X_{J_k}^\varepsilon = x_k, \forall k \leq N_t^\varepsilon, N_t^\varepsilon = \ell\} \cap F \neq \emptyset.$$

This implies that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_i(X^\varepsilon \in F_n^-) \leq - \inf_{\varphi \in F} I_t(\varphi).$$

Finally, using Lemma 7.5, we obtain

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_i(X^\varepsilon \in F) \leq \max \left(- \inf_{\varphi \in F} I_t(\varphi), -C_{n,t} \right).$$

The result is now obtained by sending n to infinity. \square

The next theorem corresponds to Theorem 4.2 in the discrete case situation. Unfortunately, as seen above, I_T is not a good rate function. This prevents us from applying directly Varadhan's lemma and leads to substantial difficulties. This is the place where we need the assumption (7.2) on \mathfrak{I} . This result makes use of the sequence $(\varepsilon_k)_{k \geq 1}$ constructed in Lemma 4.1, which holds true without modification in our discrete case. To avoid heavy notations, we shall write $\mathcal{M}_s(i)$ for $\mathcal{M}_s(\{i\})$, where \mathcal{M}_s is the measure constructed in Lemma 4.1.

Theorem 7.7. *For all (t, i) in $(0, +\infty) \times E$,*

$$\begin{aligned} V(t, i) &:= \lim_{k \rightarrow \infty} \varepsilon_k \log u^{\varepsilon_k}(t, i) \\ &= \sup_{\varphi \in \mathbb{D}([0, t], E) \text{ s.t. } \varphi_0 = i} \left\{ -h(\varphi_t) + \int_0^t \sum_{j \in E} R(\varphi_s, j) \mathcal{M}_{t-s}(j) ds - \sum_{0 < s \leq t} \mathfrak{I}(\varphi_{s-}, \varphi_s) \right\}. \end{aligned} \quad (7.4)$$

Proof. Let us fix a positive time t . To avoid heavy notations, we shall write ε instead of ε_k in this proof. Since the only part of the proof of Varadhan's Lemma relying on the compactness of the level sets of the rate function is the upper bound, we restrict ourself to this bound. As in the proof of Theorem 4.2, let $a \in \mathbb{R}$ be smaller than the right-hand side of (7.4) and C satisfying $|\Phi(\varphi) - h(\varphi(t))| \leq C(t+1)$. We define K as the (non-necessarily compact) level set

$$K = \{\varphi \in \mathbb{D}([0, t], E) \mid \varphi_0 = i, I_t(\varphi) \leq C(t+1) - a\}.$$

First, as in the proof of Theorem 4.2, we deduce from the LDP that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E} \left[\exp \left(\frac{1}{\varepsilon} \left(-h_\varepsilon(X_t^\varepsilon) + \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right) \mathbf{1}_{X^\varepsilon \in K^c} \right] \\ \leq C(t+1) + \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in K^c) \leq C(t+1) - \inf_{x \in K^c} I(x) \leq a. \end{aligned} \quad (7.5)$$

Second, by the definition of $I_t(\varphi)$, we have

$$N_\varphi \leq \frac{I_t(\varphi)}{\min_{i, j \in E} \mathfrak{I}(i, j)}. \quad (7.6)$$

Hence, according to (7.6), there exists N such that for all φ in K , $N_\varphi \leq N$. Fix $\gamma > 0$. We deduce that $K = K_{N, \gamma} \cup L_{N, \gamma}$, where

$$K_{N, \gamma} = \{\varphi \in K \mid N_\varphi \leq N \text{ and } \inf_i (t_{i+1}^\varphi - t_i^\varphi) \geq \gamma\}$$

and

$$L_{N, \gamma} = \{\varphi \in K \mid N_\varphi \leq N \text{ and } \inf_i (t_{i+1}^\varphi - t_i^\varphi) < \gamma\}.$$

Let $\Delta_\beta(T)$ be the set of subdivisions of $[0, t]$ with mesh greater than β . Since, for all $\varphi \in K_{N, \gamma}$ and all $\beta < \gamma$, we have

$$\inf_{s \in \Delta_\beta(T)} \max_{1 \leq i \leq |s|} \sup_{u, v \in [s_i, s_{i+1}]} |\varphi_u - \varphi_v| = 0,$$

we deduce from Arzela-Ascoli's theorem for the Skorokhod space that $K_{N,\gamma}$ is compact.

Fix $\delta > 0$. We deduce that there exist $n \geq 1$ and $\varphi_1^\gamma, \dots, \varphi_n^\gamma \in K_{N,\gamma}$ such that

$$K_{N,\gamma} = \bigcup_{i=1}^n G_{\varphi_i^\gamma},$$

where the neighborhood $G_{\varphi_i^\gamma}$ of φ_i^γ is chosen such that, for ε small enough,

$$-h(\varphi_i^\gamma(t)) + \Phi_\varepsilon(\varphi_i^\gamma) + \delta \geq \sup_{\varphi \in G_{\varphi_i^\gamma}} -h_\varepsilon(\varphi^\gamma(t)) + \Phi_\varepsilon(\varphi). \quad (7.7)$$

Because of Lemma 7.4, we can also assume without loss of generality that I_t is constant on $G_{\varphi_i^\gamma}$ for all i . Moreover, for all i in $\{1, \dots, n\}$ and ε small enough, we have

$$\Phi(\varphi_i^\gamma) - \delta \leq \Phi_\varepsilon(\varphi_i^\gamma) \leq \Phi(\varphi_i^\gamma) + \delta. \quad (7.8)$$

Following the lines of Theorem 4.2, we obtain

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E} \left[\exp \left(\frac{1}{\varepsilon} \left(-h_\varepsilon(X_t^\varepsilon) + \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right) \mathbb{1}_{X^\varepsilon \in K_{N,\gamma}} \right] \\ & \leq \max_{1 \leq i \leq n} \left\{ -h(\varphi_i^\gamma(t)) + \int_0^t \sum_{j \in E} R(\varphi_i^\gamma(s), j) \mathcal{M}_{t-s}(j) ds + 2\delta - I_t(\varphi_i^\gamma) \right\} \\ & \leq \max_{\varphi \in K_{N,\gamma}} \left\{ -h(\varphi(t)) + \int_0^t \sum_{j \in E} R(\varphi(s), j) \mathcal{M}_{t-s}(j) ds - I_t(\varphi) \right\} + 2\delta. \end{aligned} \quad (7.9)$$

We now prove a similar inequality when X^ε lies in $L_{N,\gamma}$. We first introduce some notations : for any $\varphi \in \mathbb{D}([0, T], E)$, in the case where $t_{N_\varphi}^\varphi \leq t - \gamma$, we define

$$\Gamma^\gamma \varphi(s) = \begin{cases} \varphi(s) & \text{if } s \in [t_i^\varphi, t_{i+1}^\varphi) \text{ for } 0 \leq i \leq N_\varphi \text{ and } t_{i+1}^\varphi - t_i^\varphi \geq \gamma \\ \varphi \left(t_{\inf\{i \leq j \leq N_\varphi \mid t_{j+1}^\varphi - t_j^\varphi \geq \gamma\}}^\varphi \right) & \text{if } s \in [t_i^\varphi, t_{i+1}^\varphi) \text{ for } 0 \leq i \leq N_\varphi \text{ and } t_{i+1}^\varphi - t_i^\varphi < \gamma, \end{cases}$$

with the convention that $t_0^\varphi = 0$ and $t_{N_\varphi+1}^\varphi = t$. In the case where $t - \gamma < t_{N_\varphi}^\varphi \leq t$, we set $\Gamma^\gamma \varphi := \Gamma^\gamma \tilde{\varphi}$ where

$$\tilde{\varphi} := \begin{cases} \varphi(t) & \text{if } s \in [t - \gamma, t], \\ \varphi(s) & \text{if } s < t - \gamma. \end{cases}$$

Now, for all φ in $L_{N,\gamma}$, $\Gamma^\gamma \varphi$ lies in $K_{N,\gamma}$. Indeed, by construction, $\Gamma^\gamma \varphi$ has inter-jumps times larger than γ and, because of Hypothesis (7.3), $I_t(\Gamma^\gamma \varphi) \leq I_t(\varphi)$ since $\Gamma^\gamma \varphi$ is obtained by suppressing jumps in φ . Hence, we get

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\frac{1}{\varepsilon} \left(-h_\varepsilon(X_t^\varepsilon) + \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right) \mathbb{1}_{X^\varepsilon \in L_{N,\gamma}} \right] \\ & \leq \sum_{i=1}^n \mathbb{E} \left[\exp \left(\frac{1}{\varepsilon} \left(-h_\varepsilon(X_t^\varepsilon) + \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right) \mathbb{1}_{X^\varepsilon \in L_{N,\gamma}} \mathbb{1}_{\Gamma^\gamma X^\varepsilon \in G_{\varphi_i^\gamma}} \right]. \end{aligned}$$

For all φ in $L_{N,\gamma}$, we necessarily have $\Gamma^\gamma \varphi(t) = \varphi(t)$, hence, it follows from the definition of Γ^γ that

$$\begin{aligned} & -h_\varepsilon(\varphi(t)) + \int_0^t R(\varphi(s), v_{t-s}^\varepsilon) ds \\ &= -h_\varepsilon(\Gamma^\gamma \varphi(t)) + \int_0^t R(\Gamma^\gamma \varphi(s), v_{t-s}^\varepsilon) ds + \sum_{t_{j+1}^\varphi - t_j^\varphi < \gamma} \int_{t_j^\varphi}^{t_{j+1}^\varphi} (R(\varphi(s), v_{t-s}^\varepsilon) - R(\Gamma^\gamma \varphi(s), v_{t-s}^\varepsilon)) ds \\ &\leq -h_\varepsilon(\Gamma^\gamma \varphi(t)) + \int_0^t R(\Gamma^\gamma \varphi(s), v_{t-s}^\varepsilon) ds + 2N\gamma \|R\|_\infty. \end{aligned}$$

Using this last inequality, (7.7) and (7.8), we get

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\frac{1}{\varepsilon} \left(-h_\varepsilon(X_t^\varepsilon) + \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right) \mathbb{1}_{x \in L_{N,\gamma}} \right] \\ &\leq \sum_{i=1}^n \mathbb{E} \left[\exp \left(\frac{1}{\varepsilon} \left(-h_\varepsilon(\Gamma^\gamma X_t^\varepsilon) + \int_0^t R(\Gamma^\gamma X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right) \exp \left(\frac{2}{\varepsilon} N\gamma \|R\|_\infty \right) \mathbb{1}_{x \in L_{N,\gamma}} \mathbb{1}_{\Gamma^\gamma X^\varepsilon \in G_{\varphi_i^\gamma}} \right] \\ &\leq \sum_{i=1}^n \exp \left(\frac{1}{\varepsilon} \left(-h(\varphi_i^\gamma(t)) + \int_0^t \sum_{j \in E} R(\varphi_i^\gamma(s), j) \mathcal{M}_{t-s}(j) ds + 2\delta \right) \right) \\ &\quad \exp \left(\frac{2}{\varepsilon} N\gamma \|R\|_\infty \right) \mathbb{P} \left(X^\varepsilon \in L_{N,\gamma}, \Gamma^\delta X^\varepsilon \in G_{\varphi_i^\gamma} \right). \end{aligned}$$

It follows from Theorem 7.6 that

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E} \left[\exp \left(\frac{1}{\varepsilon} \left(-h_\varepsilon(X_t^\varepsilon) + \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right) \mathbb{1}_{X^\varepsilon \in L_{N,\gamma}} \right] \\ &\leq \max_{1 \leq i \leq n} \left(-h(\varphi_i^\gamma(t)) + \int_0^t \sum_{j \in E} R(\varphi_i^\gamma(s), j) \mathcal{M}_{t-s}(j) ds + 2\delta + 2N\gamma \|R\|_\infty - \inf_{\varphi \in A_{\varphi_i^\gamma}} I_t(\varphi) \right) \end{aligned}$$

(with the convention $\inf_{\varphi \in \emptyset} I_t(\varphi) = +\infty$), with

$$A_{\varphi_i^\gamma} = \{\varphi \in L_{N,\gamma} \mid \Gamma^\gamma \varphi \in G_{\varphi_i^\gamma}\}.$$

Now, using Lemma 7.4 and $I_t(\Gamma^\gamma \varphi) \leq I_t(\varphi)$, we have for all i such that $A_{\varphi_i^\gamma} \neq \emptyset$,

$$I_t(\varphi_i^\gamma) = \inf_{\varphi \in A_{\varphi_i^\gamma}} I_t(\Gamma^\gamma \varphi) \leq \inf_{\varphi \in A_{\varphi_i^\gamma}} I_t(\varphi),$$

which gives

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E} \left[\exp \left(\frac{1}{\varepsilon} \left(-h_\varepsilon(X_t^\varepsilon) + \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right) \mathbb{1}_{X^\varepsilon \in L_{N,\gamma}} \right] \\ &\leq \max_{1 \leq i \leq n} \left(-h_\varepsilon(\varphi_i^\gamma(t)) + \int_0^t \sum_{j \in E} R(\varphi_i^\gamma(s), j) \mathcal{M}_{t-s}(j) ds + 2\delta + 2N\gamma \|R\|_\infty - I_t(\varphi_i^\gamma) \right) \\ &\leq \max_{\varphi \in K_{N,\gamma}} \left\{ -h(\varphi(t)) + \int_0^t \sum_{j \in E} R(\varphi(s), j) \mathcal{M}_{t-s}(j) ds - I_t(\varphi) \right\} + 2\delta + 2N\gamma \|R\|_\infty. \end{aligned}$$

Combining the last inequality with (7.5) and (7.9), we obtain

$$\begin{aligned}
& \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E} \left[\exp \left(\frac{1}{\varepsilon} \left(-h_\varepsilon(X_t^\varepsilon) + \int_0^t R(X_s^\varepsilon, v_{t-s}^\varepsilon) ds \right) \right) \right] \\
& \leq \max \left\{ a; \max_{\varphi \in K_{N,\gamma}} \left[-h(\varphi(t)) + \int_0^t \sum_{j \in E} R(\varphi(s), j) \mathcal{M}_{t-s}(j) ds - I_t(\varphi) \right] + 2\delta + 2N\gamma \|R\|_\infty \right\} \\
& \leq \sup_{\varphi \in \mathbb{D}([0,t],E) \text{ s.t. } \varphi_0=i} \left\{ -h(\varphi(t)) + \int_0^t \sum_{j \in E} R(\varphi(s), j) \mathcal{M}_{t-s}(j) ds - I_t(\varphi) \right\} + 2\delta + 2N\gamma \|R\|_\infty.
\end{aligned}$$

Since δ and γ are arbitrary, we have proved Theorem 7.7. \square

Our goal is now to obtain a version of Theorem 4.3. Since its proof makes use of the translation invariance of Brownian motion, we cannot use the same method. In particular, we will see that the function $t \mapsto \varepsilon \log u^\varepsilon(t, i)$ may not be uniformly Lipschitz for particular initial conditions. Hence we shall prove directly the Lipschitz regularity of the limit V . For this, we first need the following lemma.

Lemma 7.8. *For all subsequence $(\varepsilon_k)_{k \geq 1}$ as in Theorem 7.7, the limit $V(t, i)$ of $\varepsilon_k \log u_k^\varepsilon(t, i)$ satisfies, for all $t > 0$ and all $i \neq j \in E$,*

$$V(t, i) \geq V(t, j) - \mathfrak{I}(i, j).$$

In particular, this inequality is satisfied for all $t \geq 0$ if and only if $h(i) \leq h(j) + \mathfrak{I}(i, j)$ for all $i \neq j$.

Proof. Fix $i \neq j$, $t > 0$ and $\eta > 0$. Because of (7.4), we can choose a function $\hat{\varphi} \in \mathbb{D}([0, t], E)$ such that $\hat{\varphi}_0 = j$ and

$$V(t, j) \leq \eta - h(\hat{\varphi}_t) + \int_0^t \sum_{k \in E} R(\hat{\varphi}_s, k) \mathcal{M}_{t-s}(k) ds - \sum_{0 < s \leq t} \mathfrak{I}(\hat{\varphi}_{s-}, \hat{\varphi}_s).$$

For all $n \in \mathbb{N}$ such that $\frac{1}{n} < t_1^{\hat{\varphi}} \wedge t$, we define $\varphi^{(n)} \in \mathbb{D}([0, t], E)$ as

$$\varphi_s^{(n)} = \begin{cases} i & \text{if } 0 \leq s < 1/n \\ \hat{\varphi}_s & \text{if } 1/n \leq s \leq t. \end{cases}$$

Using (7.4) again, we have

$$\begin{aligned}
\mathfrak{I}(i, j) + V(t, i) & \geq V(t, \varphi_t^{(n)}) + \int_0^t \sum_{k \in E} R(\varphi_s^{(n)}, k) \mathcal{M}_{t-s}(j) ds - \sum_{1/n < s \leq t} \mathfrak{I}(\varphi_{s-}^{(n)}, \varphi_s^{(n)}) \\
& \geq V(t, \hat{\varphi}_t) - \frac{\|R\|_{\mathcal{H}}}{n} + \int_{1/n}^t \sum_{k \in E} R(\hat{\varphi}_s, k) \mathcal{M}_{t-s}(j) ds - \sum_{0 < s \leq t} \mathfrak{I}(\hat{\varphi}_{s-}, \hat{\varphi}_s) \\
& \geq V(t, j) - \eta - 2 \frac{\|R\|_{\mathcal{H}}}{n},
\end{aligned}$$

where $\|R\|_{\mathcal{H}} := \sup_{i \in E, x \in \mathcal{H}} |R(i, x)|$. This concludes the proof letting $\eta \rightarrow 0$ and $n \rightarrow +\infty$. \square

We can now state our result on the regularity of V .

Theorem 7.9. *For all subsequence $(\varepsilon_k)_{k \geq 1}$ as in Theorem 7.7, the limit $V(t, i)$ of $\varepsilon \log u^\varepsilon(t, i)$ is Lipschitz with respect to the time variable t on $(0, +\infty)$. In addition, if $h(i) \leq h(j) + \mathfrak{T}(i, j)$ for all $i \neq j$, the function V is Lipschitz on \mathbb{R}_+ .*

Proof. Fix $t \geq 0$ and $i \in E$. For all $\delta > 0$ and $\varepsilon > 0$, proceeding as in the proof of Theorem 4.3, Markov's property entails

$$e^{-\frac{M\delta}{\varepsilon}} \mathbb{E}_i[u^\varepsilon(t, X_\delta^\varepsilon)] \leq u^\varepsilon(t + \delta, i) \leq e^{\frac{M\delta}{\varepsilon}} \mathbb{E}_i[u^\varepsilon(t, X_\delta^\varepsilon)].$$

We can now estimate the distribution of X_δ^ε as follows. Let N^ε be the number of jumps of X^ε on the time interval $[0, \delta]$. For all $j \neq i$,

$$\delta e^{-|E|\delta} e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}} \leq \mathbb{P}_i(X_\delta^\varepsilon = j, N^\varepsilon \leq 1) = \int_0^\delta e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}} e^{-c^\varepsilon(i)s} e^{-c^\varepsilon(j)(\delta-s)} ds \leq \delta e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}}$$

where $|E|$ is the cardinality of E . Similarly,

$$1 - |E|\delta \leq \mathbb{P}_i(X_\delta^\varepsilon = i, N^\varepsilon = 0) \leq 1$$

and

$$0 \leq \mathbb{P}_i(N^\varepsilon \leq 2) \leq \sum_{j \neq i} c_\varepsilon(j) e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}} \int_0^\delta (\delta - s) ds \leq \frac{|E|^2}{2} \delta^2.$$

Putting all these inequalities together, we obtain

$$\begin{aligned} (1 - |E|\delta) u^\varepsilon(t, i) + \sum_{j \neq i} \delta e^{-|E|\delta} e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}} u^\varepsilon(t, j) &\leq \mathbb{E}_i[u^\varepsilon(t, X_\delta^\varepsilon)] \\ &\leq u^\varepsilon(t, i) + \sum_{j \neq i} \delta e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}} u^\varepsilon(t, j) + \delta^2 \bar{C} \frac{|E|^2}{2}, \end{aligned}$$

where \bar{C} is the right-hand side of the main result in Lemma 7.1. Taking $\varepsilon \log$ of both sides of the last inequality and sending ε to 0, we obtain

$$\max \left\{ V(t, i); \max_{j \neq i} V(t, j) - \mathfrak{T}(i, j) \right\} - M\delta \leq V(t + \delta, i) \leq \max \left\{ V(t, i); \max_{j \neq i} V(t, j) - \mathfrak{T}(i, j) \right\} + M\delta. \quad (7.10)$$

Since $\delta > 0$ was arbitrary, the result follows from Lemma 7.8. \square

8 Study of the variational problem in the finite case

Assume as above that E is a finite set. Our goal here is to study the limit problem

$$V(t, i) = \sup_{\varphi(0)=i} \left\{ -h(\varphi(t)) + \int_0^t \int_{\mathbb{R}^r} R(\varphi(u), y) \mathcal{M}_{t-u}(dy) du - I_t(\varphi) \right\}, \quad (t, i) \in \mathbb{R}_+ \times E. \quad (8.1)$$

A direct adaptation of Theorem 7.7 to obtain the dynamic programming version of (8.1): for any $0 \leq s \leq t$, the limit of $\varepsilon_k \log u_{\varepsilon_k}$ satisfies

$$V(t, i) = \sup_{\varphi(s)=i} \left\{ V(s, \varphi(t)) + \int_s^t \int_{\mathbb{R}^r} R(\varphi(u), y) \mathcal{M}_{s+t-u}(dy) du - I_{s,t}(\varphi) \right\},$$

where, for all φ in $\mathbb{D}([s, t], E)$

$$I_{s,t}(\varphi) = \sum_{u \in (s,t]} \mathfrak{T}(\varphi(u-), \varphi(u)).$$

In the sequel, we make use of the following assumptions on the dynamical systems related to problem (7.1). For all $A \subset E$, we define the dynamical system in $\mathbb{R}_+^A := \{(u_i, i \in A) : u_i \geq 0 \text{ for all } i \in A\}$ denoted S_A by

$$u_i = u_i R_i \left(\sum_{j \in A} \Psi_l(j) u_j, 1 \leq l \leq r \right), \quad i \in A, \quad (8.2)$$

where $R_i(x)$ stands for $R(i, x)$.

Hypothesis (H)

For all $A \subset E$, let Eq_A be the set of steady states of S_A . Assume that all element of Eq_A are hyperbolic steady states and S_A admits a unique steady state

$$u_A^* = (u_{A,i}^*)_{i \in A},$$

such that, for all i in A

$$u_{A,i}^* = 0 \quad \implies \quad R_i \left(\sum_{j \in E} \Psi_l(j) u_{A,j}^*, 1 \leq l \leq r \right) < 0.$$

Assume also that there exists a strict Lyapunov function $L_A : \mathbb{R}_+^A \rightarrow \mathbb{R}$ for the dynamical system S_A , which means that L_A is C^1 , admits as unique global minimizer on \mathbb{R}_+^A and satisfies, for any solution $u(t)$ to S_A ,

$$\frac{dL_A(u(t))}{dt} = \sum_{i \in A} \frac{\partial L_A}{\partial u_i}(u(t)) u_i(t) R_i \left(\sum_{j \in A} \Psi_l(j) u_j(A), 1 \leq l \leq r \right) < 0.$$

for all $t \geq 0$ such that $u(t) \notin \text{Eq}_A$.

The hyperbolicity assumption means that for all steady state u^* , $u_i^* = 0$ implies that

$$R_i \left(\sum_{j \in E} \Psi_l(j) u_j^*, 1 \leq l \leq r \right) \neq 0.$$

For finite dimensional dynamical systems, it is well known that the hyperbolicity condition is generic under perturbation (see [12]). Since hyperbolic equilibria are isolated, Hypothesis (H) implies that

Eq_A is finite. The global minimizer of L_A is necessarily a stable steady state of S_A , hence it must be u_A^* (since all the other steady states are not stable). In addition, the equilibrium u_A^* is globally asymptotically stable, in the sense that, for all initial condition $u(0) = (u_i(0))_{i \in A}$ in $(0, +\infty)^A$, the solution to (8.2) converges to u_A^* . Indeed, since \dot{u}_i/u_i is uniformly bounded, $u(0) \in (0, +\infty)^A$ implies that $u(t) \in (0, +\infty)^A$ for all $t > 0$. Now, the existence of a strict Lyapunov function implies that $u(t)$ converges to an equilibrium u^* . If $u^* \neq u_A^*$, because of Hypothesis (H), there exists i such that $u_i^* = 0$ and $\dot{u}_i(t)/u_i(t) \geq \frac{1}{2}R_i \left(\sum_{j \in E} \Psi_l(j)u_j^*, 1 \leq l \leq r \right) > 0$ for all t large enough. This is a contradiction with the convergence of $u(t)$ to u^* .

General classes of dynamical systems satisfying Hypothesis (H) have been given in [10]. We give here two examples.

Example 1 Our first example corresponds to indirect competition for environmental resources and is an extension of the chemostat model (1.2):

$$R_i(v) = -d_i + c_i \sum_{l=1}^r \frac{\alpha_l \Psi_l(i)}{1 + v_l},$$

where d_i, c_i, α_l are positive real numbers satisfying $c_i \sum_{l=1}^r \alpha_l \Psi_l(i) > d_i$, for all i in E .

Example 2 This example corresponds to direct competition of Lotka-Volterra (or logistic) type: we assume that $E = \{1, \dots, r\}$ and

$$R_i(v) = r_i - \sum_{l=1}^r v_l,$$

with the hypothesis that there exist positive constants c_1, \dots, c_r such that

$$c_i \Psi_i(j) = c_j \Psi_j(i), \quad \forall i, j \in E,$$

and

$$\sum_{i,j \in E} x_i x_j c_i \Psi_i(j) > 0, \quad \forall x \in \mathbb{R}^r \setminus \{0\}.$$

In this example the matrix $(\Psi_i(j))_{i,j \in E}$ is interpreted as a competition matrix between the different types of individuals. The last positivity assumption is satisfied for various competition matrices [10, 7], for example, if

$$c_i \Psi_i(j) = \beta_i \beta_j e^{-\frac{|x_i - x_j|^2}{2}},$$

for any distinct x_1, \dots, x_k in \mathbb{R}^d and β in \mathbb{R}_+^r . Another example is given by

$$c_i \Psi_i(j) = \int_{\mathbb{R}^d} e^{(x_i + x_j) \cdot z} \pi(dz),$$

for all positive measure π on \mathbb{R}^d with support having non empty interior.

Hypothesis (H) implies a key property given in the next lemma.

Lemma 8.1. *Assume Hypothesis (H). For all $A \subset E$ and all $\rho > 0$ small enough, the first hitting time $t_A^*(u(0), \rho)$ of the ρ -neighborhood of u_A^* by a solution $u(t)$ to S_A satisfies*

$$t_A^*(u(0), \rho) \leq C_\rho^* (1 + \sup_{i \in A} -\log u_i(0))$$

for some constant C_ρ^* only depending on ρ .

This lemma means that, when one coordinate $u_i(0)$ of $u(0)$ is close to zero, the time needed to converge to u_A^* grows linearly with the logarithm of $(u_i(0))^{-1}$.

Proof. Let A be a non-empty subset of E and u be a solution of the dynamical system S_A . Without loss of generality, we can assume that $L_A(u_A^*) = 0$, i.e. $\min_{u \in \mathbb{R}_+^E} L_A(u) = 0$. Let \mathcal{H}_A be the set of $u \in \mathbb{R}_+^A$ such that $\bar{u} \in \mathcal{H}$, where $\bar{u} \in \mathbb{R}_+^E$ is obtained from u by setting to zero the coordinates with indices in $E \setminus A$ and the set \mathcal{H} was defined in Lemma 7.1. Let $\mathcal{U}^* = (\text{Eq}_A \cap \mathcal{H}_A) \setminus \{u_A^*\}$.

Step 1. Decrease of $L_A(u(t))$ between the visits of two neighborhoods of steady states

Let

$$\underline{d} := \min_{u^*, v^* \in \mathcal{U}^* \cup \{u_A^*\}} |u^* - v^*|$$

and $\alpha_0 > 0$ such that

$$\sum_{i \in A} \partial_i L_A(x) x_i R_i \left(\sum_{j \in A} \Psi_l(j) x_j, 1 \leq l \leq r \right) < -\alpha_0, \quad \forall x \notin \bigcup_{u^* \in \mathcal{U}^* \cup \{u_A^*\}} B(u^*, \underline{d}/4).$$

Hence, setting $\|R\|_{\mathcal{H}} := \sup_{i \in E, x \in \mathcal{H}} |R(i, x)|$, the decrease of $L_A(u(t))$ between two visits by $u(t)$ of two distinct balls $B(u^*, \underline{d}/4)$ for $u^* \in \mathcal{U}^* \cup \{u_A^*\}$ is at least $\frac{\alpha_0 \underline{d}}{2\|R\|_{\mathcal{H}}}$. Now, let $\delta < \underline{d}/4$ be small enough to have, for all $u^* \in \mathcal{U}^*$,

$$\sup_{u \in B(u^*, \delta)} L_A(u) - \inf_{u \in B(u^*, \delta)} L_A(u) < \frac{\alpha_0 \underline{d}}{2\|R\|_{\mathcal{H}}}.$$

Hence, defining for all $k \in \mathbb{N}$,

$$\mathcal{H}_A^k := \left\{ u \in \mathcal{H}_A \mid k \frac{\alpha_0 \underline{d}}{2\|R\|_{\mathcal{H}}} \leq L_A(u) < (k+1) \frac{\alpha_0 \underline{d}}{2\|R\|_{\mathcal{H}}} \right\},$$

the solution $u(t)$ can only visit at most one ball of type $B(u^*, \delta)$ for some $u^* \in \mathcal{U}^*$ during its travelling time through \mathcal{H}_A^k .

Step 2. Time spent in \mathcal{H}_A^k

According to Hypothesis (H), for any steady state $u^* \in \mathcal{U}^*$, we have

$$R_i \left(\sum_{j \in A} \Psi_l(j) u_j^*, 1 \leq l \leq r \right) > 0, \quad \text{for at least one } i(u^*) \in A \text{ such that } u_{i(u^*)}^* = 0.$$

Consequently, reducing $\delta > 0$ if necessary, there exists a positive real number R_- such that, for all $u^* \in \mathcal{U}^*$,

$$\forall x \in B(u^*, \delta), \quad R_{i(u^*)} \left(\sum_{j \in A} \Psi_l(j) u_j^*, \quad 1 \leq l \leq r \right) > R_-. \quad (8.3)$$

In addition, since L_A is a strict Lyapunov function, there exists a positive real number α such that

$$\sum_{i \in A} \partial_i L_A(x) x_i R_i \left(\sum_{j \in A} \Psi_l(j) x_j, \quad 1 \leq l \leq r \right) < -\alpha, \quad \forall x \notin \bigcup_{u^* \in \mathcal{U}^* \cup \{u_A^*\}} B(u^*, \delta). \quad (8.4)$$

Now, let c be a positive real number satisfying

$$c^{-1} < \alpha^{-1} \|R\|_{\mathcal{H}}.$$

Let k_0 such that $u_0 \in \mathcal{H}_A^{k_0}$. For any $0 \leq k \leq k_0$, let t_k be the hitting time of \mathcal{H}^k and t_{-1} the first hitting time of $B(u_A^*, \delta)$, and define for all $k \geq 0$

$$F_k(t) = \begin{cases} L_A(u(t)) & \text{if } u(t) \notin \bigcup_{u^* \in \mathcal{U}^* \cup \{u_A^*\}} B(u^*, \delta) \text{ for all } t \in [t_k, t_{k-1}], \\ L_A(u(t)) - c \log u_{i(u^*)}(t) & \text{if } \exists t \in [t_k, t_{k+1}], \exists u^* \in \mathcal{U}^* \text{ such that } u(t) \in B(u^*, \delta). \end{cases}$$

Note that Step 1 implies that, for all $k \geq 0$, there exists at most one $u^* \in \mathcal{U}^*$ such that $u(t) \in B(u^*, \delta)$ for some $t \in [t_k, t_{k+1}]$. Using (8.4), (8.3), the fact that $\frac{d}{dt} L_A(u(t)) \leq 0$ and the definition of $\|R\|_{\mathcal{H}}$, we have for all $t \in [t_k, t_{k+1}]$,

$$\frac{dF_k(t)}{dt} \leq (-cR_-) \vee (-\alpha + c\|R\|_{\mathcal{H}}) < 0.$$

Therefore, for all $t \in [t_k, t_{k+1}]$,

$$F_k(t) \leq (k+1) \frac{\alpha_0 d}{2\|R\|_{\mathcal{H}}} - c \inf_{i \in A} \log u_i(t_k).$$

In addition it follows from Lemma 7.1 that

$$F_k(t) \geq k \frac{\alpha_0 d}{2\|R\|_{\mathcal{H}}} - \log \left(\frac{v_{\max} + A(|E| - 1)e^{-\gamma/\varepsilon}}{\Psi_{\min}} \right).$$

Setting $k = k_0$, we deduce that there exists a constant $C_{k_0-1} > 0$ such that

$$t_{k_0-1} \leq C_{k_0-1} (1 - \inf_{i \in A} \log u_i(0))$$

and, since $\left| \frac{\dot{u}_i}{u_i} \right| \leq \|R\|_{\mathcal{H}}$,

$$-\inf_{i \in A} \log u_i(t_{k_0-1}) \leq -\inf_{i \in A} \log u_i(0) + \|R\|_{\mathcal{H}} C_{k_0-1} (1 - \inf_{i \in A} \log u_i(0)).$$

Proceeding by induction, it follows that there exist constants C_k and D_k depending only on k_0 and $u(0)$ such that, for all $k \geq 1$,

$$t_{k-1} - t_k \leq C_k(1 - \inf_{i \in A} \log u_i(0)) \quad \text{and} \quad - \inf_{i \in A} \log u_i(t_{k-1}) \leq D_k(1 - \inf_{i \in A} \log u_i(0)).$$

Similarly, there exist constants C_k and D_k such that

$$t_{-1} - t_0 \leq C_0(1 - \inf_{i \in A} \log u_i(0)) \quad \text{and} \quad - \inf_{i \in A} \log u_i(t_{-1}) \leq D_0(1 - \inf_{i \in A} \log u_i(0)).$$

Since L is a strict Lyapunov function, for all $\rho < \delta$, the time needed to enter $B(u_A^*, \rho)$ starting from any point in $B(u_A^*, \delta)$ is bounded by a constant depending only on ρ . The result follows. \square

This lemma entails the following property.

Proposition 8.2. *Assume Hypothesis (H). Let $(\varepsilon_k)_{k \geq 1}$ be as in Theorem 7.9. For any $t \geq 0$, there exists $\rho_t > 0$ such that, for all $s \in (t, t + \rho_t]$, v_s^ε converges to $F(\{V(t, \cdot) = 0\})$, where the convergence is uniform in all compact subsets of $(t, t + \rho_t]$ and where*

$$F(A) = \left(\sum_{j=1}^r \eta_i(j) u_{A,j}^* \right)_{1 \leq i \leq r}, \quad \forall A \subset E.$$

In particular, the weak limit \mathcal{M}_s of $\delta_{v_{\varepsilon_k}(s)}$ obtained in Lemma 4.1 satisfies

$$\mathcal{M}_s = \delta_{F(\{V(t, \cdot) = 0\})}, \quad \text{for almost all } s \in (t, t + \rho_t).$$

and the function $t \mapsto F(\{V(t, \cdot) = 0\})$ is right-continuous.

Note that, because the set E is finite, the range of the function F is finite, and thus, assuming that $t \mapsto F(\{V(t, \cdot) = 0\})$ is right-continuous implies that the union of the time intervals where this function is constant is equal to \mathbb{R}_+ . We emphasize that this is not true in general for measurable functions taking values in finite sets.

Proof. Let us fix $t \geq 0$ and define $A = \{i \in E \mid V(t, i) = 0\}$. Since, at time t , we have $V(t, i) = \lim_{k \rightarrow \infty} \varepsilon_k \log u^{\varepsilon_k}(t, i)$, it follows that, for all $\delta > 0$, for all k large enough,

$$u^{\varepsilon_k}(t, i) \geq e^{-\frac{\delta}{\varepsilon_k}}, \quad \forall i \in A. \tag{8.5}$$

In addition, if $t > 0$, using the continuity of V at time t (see Theorem 7.9), there exists some positive real numbers η_t and α such that $V(t + s, i) < -\alpha$ for all $i \in E \setminus A$ and all $s \in [0, \eta_t]$. If $t = 0$, setting

$$\beta = \max \left(\max_{i \in E \setminus A} -h(i), \max_{i \neq j} -h(j) - \mathfrak{T}(i, j) \right) < 0$$

and $-\alpha \in (\beta, 0)$, it follows from the upper bound of (7.10) that $V(t + s, i) \leq -\alpha$ for all $i \in E \setminus A$ and all $s \in [0, \eta_t]$ where

$$\eta_t = \frac{\beta - \alpha}{M}.$$

In both cases, we obtain

$$\sup_{s \in [0, \eta_t]} u^{\varepsilon_k}(t+s, i) \leq e^{-\frac{\alpha}{\varepsilon_k}}, \quad \forall i \in E \setminus A. \quad (8.6)$$

Now, let $(u_i(s), s \in \mathbb{R}_+)_{i \in A}$ be the solution of the dynamical system S_A , as defined in (8.2), with initial conditions $u_i(0) = u^{\varepsilon_k}(t, i)$, for all i in A . According to (8.2) and (7.1), we have, for any time s and any i in A ,

$$\begin{aligned} & |u^{\varepsilon_k}(t+s, i) - u_i(s/\varepsilon_k)| \\ & \leq \frac{1}{\varepsilon_k} \int_0^s \left| u^{\varepsilon_k}(t+u, i) R(i, v_{t+u}^{\varepsilon_k}) - u_i(u/\varepsilon_k) R_i \left(\sum_{j \in A} \Psi_l(j) u_j(u/\varepsilon_k), 1 \leq l \leq k \right) \right| du \\ & \quad + \int_0^s \sum_{j \in E} e^{-\frac{\mathfrak{I}(i,j)}{\varepsilon_k}} |u^{\varepsilon_k}(u, j) - u^{\varepsilon_k}(u, i)| du. \end{aligned}$$

Using Hypothesis (2d), in conjunction with Lemma 7.1, we get

$$|u^{\varepsilon_k}(t+s, i) - u_i(s/\varepsilon_k)| \leq \frac{D}{\varepsilon_k} \int_0^s |u^{\varepsilon_k}(t+u, i) - u_i(u/\varepsilon_k)| du + 2Cse^{-\frac{\gamma}{\varepsilon_k}},$$

with γ defined in Lemma 7.1 and some positive constants C and D . Hence, Gronwall Lemma entails that

$$|u^{\varepsilon_k}(t+s, i) - u_i(s/\varepsilon_k)| \leq \frac{\tilde{C}}{\varepsilon_k} \exp\left(\frac{Ds - \gamma}{\varepsilon_k}\right), \quad (8.7)$$

for some positive constant \tilde{C} .

Now, let us compare $u^{\varepsilon_k}(t+s, i)$ with the expected limit $u_{i,A}^*$ which gives

$$|u^{\varepsilon_k}(t+s, i) - u_{i,A}^*| \leq |u^{\varepsilon_k}(t+s, i) - u_i(s/\varepsilon_k)| + |u_i(s/\varepsilon_k) - u_{i,A}^*|.$$

According to Lemma 8.1, we have, for any positive real number ρ ,

$$\frac{s}{\varepsilon_k} > C_\rho^*(1 + \sup_{i \in A} -\log u_i(0)) \implies |u_i(s/\varepsilon_k) - u_{i,A}^*| < \rho.$$

However, according to (8.5), for all $\delta > 0$, for k large enough,

$$-\log u_i(0) \leq \frac{\delta}{\varepsilon_k}, \quad \forall i \in A..$$

This last inequality gives, for all k large enough,

$$s > 2\delta C_\rho^* \implies |u_i(s/\varepsilon_k) - u_{i,A}^*| < \rho.$$

This entails in conjunction with (8.7) that

$$\limsup_{k \rightarrow +\infty} \sup_{s \in [2\delta C_\rho^*, \gamma/2D]} |u^{\varepsilon_k}(t+s, i) - u_{i,A}^*| < \rho, \quad \forall i \in A.$$

Since ρ and δ were arbitrary, the desired uniform convergence follows from (8.6) with $\rho_t = \frac{\gamma}{2D} \wedge \eta_t$. The remaining statements follow easily. \square

Corollary 8.3. *Assume Hypothesis (H). Any limit V of $\varepsilon_k \log u^{\varepsilon_k}$ along a subsequence as in Theorem 7.9 satisfies $V(0, i) = -h(i)$ for all $i \in E$ and for all $t \geq 0$*

$$V(t, i) = \sup_{\varphi(0)=i} \left\{ -h(\varphi(t)) + \int_0^t R(\varphi(u), F(\{V(t-u, \cdot) = 0\})) du - I_t(\varphi) \right\},$$

and its dynamic programming version

$$V(t, i) = \sup_{\varphi(s)=i} \left\{ V(s, \varphi(t)) + \int_s^t R(\varphi(u), F(\{V(t+s-u, \cdot) = 0\})) du - I_{s,t}(\varphi) \right\}. \quad (8.8)$$

In addition, the problem (8.8) admits a unique solution such that $t \mapsto F(\{V(t, \cdot) = 0\})$ is right-continuous. In particular, the full sequence $(\varepsilon \log u^\varepsilon)_{\varepsilon>0}$ converges to this unique solution when $\varepsilon \rightarrow 0$.

Proof. The only non-obvious consequence of Proposition 8.2 is the uniqueness of a solution to (8.8) with $t \mapsto F(\{V(t, \cdot) = 0\})$ right-continuous. To prove this, observe that, by continuity of V for $t > 0$ and using (7.10) for $t = 0$, the set

$$A = \{t \geq 0 \mid \text{there exists a unique solution up to time } t\}$$

cannot have a finite upper bound. □

We conclude by giving the discrete Hamilton-Jacobi formulation of the variational problem solved by V .

Theorem 8.4. *Under Hypothesis (H) and assuming that $h(i) \leq h(j) + \mathfrak{T}(i, j)$ for all $i \neq j$, the problem*

$$\dot{V}(t, i) = \sup \{R_j(F(\{V(t, \cdot) = 0\})) \mid j \in E, V(t, j) - \mathfrak{T}(j, i) = V(t, i)\}, \quad V(0, i) = h(i) \quad \forall i \in E \quad (8.9)$$

(with the convention $\mathfrak{T}(i, i) = 0$) admits a unique solution such that $t \mapsto F(\{V(t, \cdot) = 0\})$ is right-continuous which is the unique solution to the variational problem (8.8).

The result also extends to cases where the condition $h(i) \leq h(j) + \mathfrak{T}(i, j)$ is not satisfied for some $i \neq j$. In this case, one must replace the initial condition in (8.9) by

$$V(0, i) = \max \left\{ -h(i); \max_{j \neq i} -h(j) - \mathfrak{T}(i, j) \right\}, \quad \forall i \in E$$

and the solution of (8.9) coincides with the solution to the variational problem (8.8) for positive times only.

Proof. We first prove the uniqueness part and then that the solution of (8.8) also solves (8.9).

Step 1: Uniqueness for (8.9).

Let $t \in [0, +\infty]$ be the largest time such that there is uniqueness for (8.9) up to time t and assume $t < \infty$. Let $i \in E$ be such that $V(t, i) = 0$. Since $\mathfrak{T}(j, i) > 0$ for all $j \neq i$ and because of the Lipschitz regularity of any solution V of (8.9), $\dot{V}(s, i) = R_i(F(\{V(t, \cdot) = 0\}))$ for all $s \in [t, t + \delta_t]$ for

some $\delta_t > 0$. Hence $V(s, i)$ is uniquely determined for all i such that $V(t, i) = 0$ and $s \in [t, t + \delta_t]$. We proceed similarly for all the $i' \in E$ such that $V(t, i') \in [-\inf_{i \neq j} \mathfrak{T}(i, j)/2, 0)$: their dynamics is determined only by $F(\{V(t, \cdot) = 0\})$ and $V(s, i)$ for all i such that $V(t, i) = 0$, for $s \geq t$ in the time interval $[t, t + \delta_t]$. We obtain a similar result inductively for all the $i' \in E$ such that $V(t, i') \in [-k \inf_{i \neq j} \mathfrak{T}(i, j)/2, -(k-1) \inf_{i \neq j} \mathfrak{T}(i, j)/2)$ for all $k \geq 1$. This contradicts the finiteness of t and hence uniqueness for (8.9) is proved.

Step 2: The function V of Corollary 8.3 solves (8.9).

Let $t \geq 0$ and $i \in E$ be fixed and let us prove that the solution V of (8.8) is right-differentiable with respect to time at (t, i) with derivative given by (8.9). According to Corollary 8.3, we have for all $\delta > 0$

$$V(t + \delta, i) = \sup_{\varphi(t)=i} \left\{ V(t, \varphi(t + \delta)) + \int_t^{t+\delta} R(\varphi(u), F(\{V(t + \delta - u, \cdot) = 0\})) du - I_{t, t+\delta}(\varphi) \right\} \quad (8.10)$$

and there exists $\delta_t > 0$, such that $F(\{V(t + \delta, \cdot) = 0\}) = F(\{V(t, \cdot) = 0\})$ for all $\delta < \delta_t$.

We know from Lemma 7.8 that $V(t, i) \geq V(t, j) - \mathfrak{T}(i, j)$ for all $i \neq j$. If the inequality is strict for all $j \neq i$, it is clear that the supremum in (8.10) is attained for the constant function $\varphi \equiv i$ for sufficiently small $\delta > 0$ and hence $\dot{V}(t + \delta, i) = R_i(F(\{V(t, \cdot) = 0\}))$, which entails (8.9).

If there exists $j \neq i$ such that $V(t, i) = V(t, j) - \mathfrak{T}(i, j)$, let $j^* \in E$ be such that $R_j(F(\{V(t, \cdot) = 0\}))$ is maximal among all $j \in E$ such that $V(t, i) = V(t, j) - \mathfrak{T}(i, j)$. Since the supremum in (8.10) cannot be attained for functions φ jumping two times or more in $[t, t + \delta]$ for $\delta > 0$ small enough, considering all possible choices for φ with less than one jump, one easily checks that, for $\delta > 0$ small enough,

$$\begin{aligned} V(t + \delta, i) &= \lim_{n \rightarrow +\infty} V(t, \varphi^{(n)}(t + \delta)) + \int_t^{t+\delta} R(\varphi^{(n)}(u), F(\{V(t, \cdot) = 0\})) du - \mathfrak{T}(i, j^*) \\ &= V(t, j^*) - \mathfrak{T}(i, j^*) + \delta R_{j^*}(F(\{V(t, \cdot) = 0\})) \\ &= V(t, i) + \delta R_{j^*}(F(\{V(t, \cdot) = 0\})) \end{aligned}$$

where $\varphi_{t+s}^{(n)} = i$ if $s < 1/n$ and $\varphi_{t+s}^{(n)} = j^*$ if $1/n \leq s \leq \delta$. Hence $\dot{V}(t + \delta, i) = R_{j^*}(F(\{V(t, \cdot) = 0\}))$, which gives the result. \square

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