# Quasi-stationary distributions in reducible state spaces

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**Abstract**

We study quasi-stationary distributions and quasi-limiting behavior of Markov chains in general reducible state spaces with absorption. We propose a set of assumptions dealing with particular situations where the state space can be decomposed into three subsets between which communication is only possible in a single direction. These assumptions allow us to characterize the exponential order of magnitude and the exact polynomial correction, called polynomial convergence parameter, for the leading order term of the semigroup for large time. They also provide explicit convergence speeds to this leading order term. We apply these results to general Markov chains with finitely or denumerably many communication classes using a specific induction over the communication classes of the chain. We are able to explicitely characterize the polynomial convergence parameter, to determine the complete set of quasi-stationary distributions and to provide explicit estimates for the speed of convergence to quasi-limiting distributions in the case of finitely many communication classes. We conclude with an application of these results to the case of denumerable state spaces, where we are able to prove that, in general, there is existence of a quasi-stationary distribution without assuming irreducibility before absorption. This actually holds true assuming only aperiodicity, the existence of a Lyapunov function and the existence of a point in the state space from which the return time is finite with positive probability.

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# 1 Introduction

Let $(X_n, n \in \mathbb{Z}_+)$ be a Markov chain in $D \cup \{\partial\}$ where $D$ is a measurable space, $\partial \not\in D$ and $\mathbb{Z}_+ := \{0, 1, \dots\}$. For all $x \in D \cup \{\partial\}$, we denote as usual by $\mathbb{P}_x$ the law of $X$ given $X_0 = x$ and for any probability measure $\mu$ on $D \cup \{\partial\}$, we define $\mathbb{P}_\mu = \int_{D\cup\{\partial\}} \mathbb{P}_x \, \mu(dx)$. We also denote by $\mathbb{E}_x$ and $\mathbb{E}_\mu$ the associated expectations. We assume that $\partial$ is absorbing, which means that $X_n = \partial$ for all $n \geq \tau_\partial$, $\mathbb{P}_x$-almost surely, where

$$\tau_\partial = \inf\{n \in \mathbb{Z}_+, \, X_n = \partial\}.$$

We study the sub-Markovian transition semigroup of $X$, $(S_n)_{n\in\mathbb{Z}_+}$, defined as

$$S_n f(x) = \mathbb{E}_x \left( f(X_n) \mathbb{1}_{n<\tau_\partial} \right), \ \forall n \in \mathbb{Z}_+,$$

for all bounded or nonnegative measurable function $f$ on $D$ and all $x \in D$. We also define as usual the left-action of $P_n$ on measures as

$$\mu S_n f = \mathbb{E}_\mu \left( f(X_n) \mathbb{1}_{n<\tau_\partial} \right) = \int_D S_n f(x) \, \mu(dx),$$

for all probability measure $\mu$ on $D$ and all bounded or nonnegative measurable function $f : D \to \mathbb{R}$.

The purpose of this article is to provide original and practical criteria allowing to study the quasi-limiting behaviour of absorbed, reducible Markov processes in general state spaces, both in cases of geometric and polynomial convergence in total variation to a quasi-stationary distribution.

We recall that a quasi-stationary distribution (QSD) for $X$ is a probability measure $\nu_{QS}$ on $D$ such that

$$\mathbb{P}_{\nu_{QS}}(X_n \in \cdot \mid n < \tau_\partial) = \frac{\nu_{QS} S_n}{\nu_{QS} S_n \mathbb{1}_D} = \nu_{QS}, \ \forall n \geq 0.$$

It is well known that a probability measure $\nu_{QS}$ is a QSD for $X$ if and only if it is a quasi-limiting distribution (see e.g. [12, 23]). By a quasi-limiting distribution $\nu$, we mean a probability measure $\nu$ such that, for some probability measure $\mu$ on $D$ and for any measurable subset $\Gamma \subset D$, the conditional probabilities $\mathbb{P}_\mu(X_n \in \Gamma \mid n < \tau_\partial)$ converges to $\nu(\Gamma)$. To each QSD $\nu_{QS}$ is associated an *exponential convergence parameter* $\theta \in (0,1]$, such that

$$\mathbb{P}_{\nu_{QS}}(\tau_\partial \geq n) = \theta^n, \ \forall n \geq 0.$$

This parameter is called a *convergence parameter* in [25], and we add the term *exponential* to distinguish it from the *polynomial convergence parameter* that we introduce below.

The study of quasi-limiting behaviour of Markov chains on reducible state spaces started with the work of Mandl [22] (see also [13]). Since then, several works studied cases of finite state spaces [26, 7, 5, 27, 28] or infinite state spaces [17, 8]. Most of these works are devoted to spectific processes, while the articles [27, 28] address the general situation in finite state spaces (see also the survey [29]).

In order to obtain results on general state spaces, we make use of results on the principal eigenvalue and eigenvectors of iterates of upper triangular matrices of linear operators over a Banach space (the developments can be found in the Appendix). This allows us to prove sufficient conditions ensuring that a reducible process $X$ satisfies

$$\left\| \theta^{-n} n^{-j(x)} \mathbb{P}_x(X_n \in \cdot) - \eta(x)\nu_{QS} \right\| \xrightarrow[n\to+\infty]{} 0, \quad \forall x \in D, \tag{1.1}$$

for some measurable functions $\eta : D \to [0,+\infty)$ and $j : D \to \mathbb{Z}_+ = \{0,1,\dots\}$, and where $\|\cdot\|$ is a weighted total variation norm (see Assumption (A) in Section 2 for more details). We call the function $j$ the *polynomial convergence parameter*. and prove several properties of $j$, $\eta$ and $\nu_{QS}$ in Section 2.

We emphasize that for many usual irreducible Markov processes, the quasi-limiting behaviour is well understood and it is known that this result holds true with $j \equiv 0$ (see for instance [12, 23, 9, 10]). This is also true for some reducible processes with exponential convergence (see [10, Thm 6.1]). The main novelties of this paper are firstly to consider reducible processes in general state spaces with sub-geometric convergence in (1.1) and with non-uniformly zero $j$, and secondly to provide a general methodology to obtain precise convergence results as in (1.1). Of course, all the results of the present paper can be easily extended to general unbounded semigroups (i.e. not necessarily sub-Markov) following the same approach as in [11].

The paper is organized as follows. In Section 2, we present our main assumption and its first consequences. In Section 3, we consider reducible sub-Markov processes with two successive sets where this assumption is satisfied. We then consider in Section 4 reducible sub-Markov processes with several communication classes. As an illustration, we prove in Section 5 that, under a mild Lyapunov assumption, processes on discrete state spaces admit quasi-limiting distributions. In the Appendix, we provide a detailed study of iterates of upper triangular matrix of linear operators over a Banach space.

**Notation.** The set $\mathscr{M}(D)$ is the Banach space of finite signed measures over $D$, endowed with the total variation norm. We denote by $\mathscr{M}_+(D) \subset \mathscr{M}(D)$ the set of non-negative finite measures over $D$. Given a positive measurable function $W$, the set $\mathscr{M}(W)$ is the Banach space of signed measures $\mu$ such that $|\mu|(W) < +\infty$, endowed with the norm

$$\|\mu\|_W := |\mu|(W).$$

We extend the operator $S_n$ to $\mathscr{M}(D)$ by $\mu S_n = \int_D \delta_x S_n \mu(\mathrm{d}x)$. The set $L^\infty(W)$ is the Banach space of measurable functions $f$ such that $\|f/W\|_\infty < +\infty$, endowed with norm

$$\|f\|_W := \|f/W\|_\infty.$$

Because of the nature of our problem, we will often consider the extensions to $D \cup \{\partial\}$ of functions defined on a subset of $D \cup \{\partial\}$. Systematically and without further notice, all functions are extended by the value 0 outside of their domain of definition. In all the sequel, $C$ will denote a finite constant that may change from line to line.

# 2 Exponential and polynomial convergence parameter

We consider a discrete time Markov process $(X_n, n \in \mathbb{Z}_+)$ evolving in a measurable set $D \cup \{\partial\}$ with absorption at $\partial \notin D$ at time $\tau_\partial$, and sub-Markovian semigroup $(S_n)_{n \in \mathbb{Z}_+}$. We recall that the *exponential convergence parameter* of the semigroup $(S_n)_{n \in \mathbb{N}}$ is given by

$$\theta_S(\mu) := \inf\left\{\theta \geq 0, \liminf_{n \to +\infty} \theta^{-n} \mu S_n \mathbb{1}_D = 0\right\}, \ \forall \mu \in \mathscr{M}_+(D).$$

We also set $\theta_{0,S} = \sup_{x\in D}\theta_S(x)$, where $\theta_S(x) = \theta_S(\delta_x)$. We define the *polynomial convergence parameter* of the semigroup $(S_n)_{n\in\mathbb{N}}$ as

$$j_S(\mu) := \inf\{\ell \geq 0,\ \liminf_{n\to+\infty} n^{-\ell}\theta_{0,S}^{-n}\mu S_n \mathbb{1}_D = 0\},\ \forall \mu \in \mathcal{M}_+(D) \tag{2.1}$$

with the convention $\inf\emptyset = 0$. We also set $j_{0,S} = \sup_x j_S(x)$, where $j_S(x) := j_S(\delta_x)$. Note that if $\theta_S(\mu) < \theta_{0,S}$, then $j_S(\mu) = 0$. We will see in Proposition 2.1 below that the converse inequality $\theta_S(\mu) > \theta_{0,S}$ never happens.

In this section, we are interested in the implications of the following assumption (A) on $j_S$ and on the existence and convergence toward a quasi-stationary distribution for $X$. In the following sections, we will study sufficient properties implying that $X$ satisfies this condition.

**Assumption (A).** We have $\theta_{0,S} \in (0,1]$, $j_S$ is integer valued and there exist a measurable function $W_S : D \to [1,+\infty)$, a finite or countable set $I_S$ and some probability measures $\nu_{S,i} \in \mathcal{M}(W_S)$ and non-identically zero non-negative $\eta_{S,i} \in L^\infty(W_S)$ for each $i \in I_S$, such that

$$\sum_{i\in I_S} \eta_{S,i}\nu_{S,i}(W_S) \in L^\infty(W_S) \tag{2.2}$$

and such that, for all $f \in L^\infty(W_S)$, all $n \geq 1$ and all $x \in D$,

$$\left|\theta_{0,S}^{-n} n^{-j_S(x)}\mathbb{E}_x(f(X_n)\mathbb{1}_{n<\tau_\partial}) - \sum_{i\in I_S}\eta_{S,i}(x)\nu_{S,i}(f)\right| \leq \alpha_{S,n} W_S(x)\|f\|_{W_S}, \tag{2.3}$$

where $\alpha_{S,n}$ goes to 0 when $n \to +\infty$.

When Assumption (A) holds true, we define

$$\eta_S := \sum_{i\in I_S}\eta_{S,i} \in L^\infty(W_S). \tag{2.4}$$

Note that (2.3) only gives an equivalent of $\delta_x S_n \mathbb{1}_D$ when $\eta_{S,i}(x) > 0$ for at least one $i \in I_S$. In particular, for all $x$ such that $\theta_S(x) < \theta_{S,0}$, (2.3) implies that $\eta_{S,i}(x) = 0$ for all $i \in I_S$.

We also emphasize that, for all $x \in D$ such that $\eta_S(x) > 0$, (2.3) entails that $\mathbb{P}_x(X_n \in \cdot \mid n < \tau_\partial)$ converges in $\mathcal{M}(W_S)$ toward $\frac{1}{\eta_S(x)}\sum_{i\in I_S}\eta_{S,i}(x)\nu_{S,i}$, which is thus a quasi-limiting distribution and hence a quasi-stationary distribution. The rest of this section is dedicated to the exposition and proofs of finer properties on $j_S$ and on the quasi-stationary distributions of $X$ under Assumption (A).

*Remark* 1. The results of this section, after minor modifications, hold true without assuming that $j_S$ is integer-valued in Assumption (A). However, assuming $j_S$ integer valued simplifies several expressions and allows us to obtain explicit speed of convergence toward a quasi-stationary distribution (see in particular Remark 6 below). In addition, in the next sections, we prove that $j_S$ is integer valued for the class of reducible processes we are interested in. We refer to[10, Section 3.2] were similar arguments are developed in a more complicated context, but with $j_S \equiv 0$. △

*Remark* 2. For simplicity, we use a countable set $I_S$ in Assumption (A). Most of the results below can be extended without too much effort to a non-countable measured set $(I_S, \mathscr{I}_S, \xi)$, but the exposition of the assumptions and results become a little bit more technical. In particular, $\sum_{i\in I_S} \eta_{S,i} \nu_{S,i}$ would be replaced by

$$\int_{I_S} \eta_{S,i} \nu_{S,i} \,\mathrm{d}i,$$

with appropriate measurability conditions. Since our setting is complicated enough, we leave the details of the adaptation to the interested reader. △

*Remark* 3. The results of this paper are stated in the discrete-time setting. The adaptation to the continuous time setting can be obtained by considering Assumption (A) for the included Markov chain and by assuming in addition that, for all $t \in [0,1]$, $\mathbb{E}_x(W_S(X_t)) \le C W_S(x)$ for some constant $C > 0$. △

We start our study with simple properties on the polynomial convergence parameter $j_S$.

**Proposition 2.1.** *For all $\mu \in \mathscr{M}_+(D)$,*

$$\theta_S(\mu) \ge \sup\left\{\theta \ge 0,\ \mu\{x,\ \theta_S(x) \ge \theta\} > 0\right\} \tag{2.5}$$

*and*

$$j_S(\mu) \ge \sup\left\{\ell \ge 0,\ \mu\{x,\ j_S(x) \ge \ell\} > 0\right\} \tag{2.6}$$

*If Assumption (A) holds true, then $j_S$ is lower semi-continuous on $\mathscr{M}_+(W_S)$ and, for all $\mu \in \mathscr{M}_+(W_S)$,*

$$\theta_S(\mu) = \sup\left\{\theta \ge 0,\ \mu\{x,\ \theta_S(x) \ge \theta\} > 0\right\} \tag{2.7}$$

*and*

$$j_S(\mu) = \sup\Big\{\ell \geq 0,\ \mu\{x,\ j_S(x) \geq \ell\} > 0\Big\}. \tag{2.8}$$

*In addition,*

$$j_S(\mu) = j_S(\mu S_1). \tag{2.9}$$

*In particular, $(j_S(X_n))_{n\geq 0}$ is $\mathbb{P}_x$-almost surely non-increasing, for all $x \in D$.*

*Remark* 4. The proof of this proposition does not use the fact that $j_S$ is assumed integer-valued in Assumption (A). In particular, this result implies that, under Assumption (A) without the property that $j_S$ is integer valued, the function $j_S$ is integer-valued over $\mathscr{M}_+(W_S)$ if and only if $j_S(x)$ in an integer for all $x \in D$. △

*Proof of Proposition 2.1.* We prove (2.6), (2.8) and (2.9) in this order. The proof of (2.5), (2.7) are similar and thus omitted.

*Proof of* (2.6). Fix a positive measure $\mu$ on $D$ (the result is trivial if $\mu = 0$). For all $\varepsilon > 0$ and for all $x \in D$ such that $j_S(\mu) + \varepsilon < j_S(x)$, we have by definition of $j_S(x)$ and the fact that $(j_S(\mu) + \varepsilon + j_S(x))/2 < j_S(x)$,

$$\liminf_{n\to+\infty} \theta_{0,S}^{-n} n^{-(j_S(\mu)+\varepsilon+j_S(x))/2} \delta_x S^n \mathbb{1}_D > 0$$

and hence, since $(j_S(\mu) + \varepsilon + j_S(x))/2 > j_S(\mu) + \varepsilon$,

$$\liminf_{n\to+\infty} \theta_{0,S}^{-n} n^{-j_S(\mu)-\varepsilon} \delta_x S^n \mathbb{1}_D = +\infty.$$

Using Fatou's Lemma, we obtain

$$\begin{aligned} 0 = \liminf_{n\to+\infty} \theta_{0,S}^{-n} n^{-j_S(\mu)-\varepsilon} \mu S^n \mathbb{1}_D &\geq \mu\Big(\liminf_{n\to+\infty} \theta_{0,S}^{-n} n^{-j_S(\mu)-\varepsilon} S^n \mathbb{1}_D\Big) \\ &\geq \mu\Big(+\infty \mathbb{1}_{j_S(\cdot)>j_S(\mu)+\varepsilon}\Big). \end{aligned}$$

This implies that, for all $\varepsilon > 0$, $\mu\{x,\ j_S(x) > j_S(\mu)+\varepsilon\} = 0$, and hence that $\mu\{x,\ j_S(x) > j_S(\mu)\} = 0$. In particular, any $\ell \geq 0$ such that $\mu\{x, j_S(x) \geq \ell\} > 0$ satisfies $\ell \leq j_S(\mu)$. We thus proved that

$$j_S(\mu) \geq \ell_\mu := \sup\Big\{\ell \geq 0,\ \mu\{x,\ j_S(x) \geq \ell\} > 0\Big\}. \tag{2.10}$$

*Proof of* (2.8) *and the fact that $j_S$ is lower semi-continuous.* We assume that $\mu \in \mathscr{M}_+(W_S)$ is a positive measure and that Assumption (A) holds true, and we prove $j_S(\mu) \leq \ell_\mu$, where $\ell_\mu$ is defined in (2.10). Fix $\ell > \ell_\mu$, so $j_S(x) < \ell$ $\mu(\mathrm{d}x)$-almost everywhere. Then, by Assumption (A),

$$\begin{aligned}\left|\theta_{0,S}^{-n} n^{-\ell} \delta_x S^n \mathbb{1}_D\right| &\leq n^{-(\ell - j_S(x))}\left(\alpha_{S,n} W_S(x) + \eta_S(x)\right)\\ &\leq n^{-(\ell - j_S(x))} C\, W_S(x)\\ &\xrightarrow[n\to+\infty]{\mu(\mathrm{d}x)-\text{a.e.}} 0\end{aligned}$$

for some constant $C > 0$. This also implies that $\left|\theta_{0,S}^{-n} n^{-\ell} \delta_x S^n \mathbb{1}_D\right|$ is bounded, up to a multiplicative constant, by the $\mu$-integrable function $W_S$, and hence, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} \theta_{0,S}^{-n} n^{-\ell} \mu S^n \mathbb{1}_D = 0.$$

This entails that $\ell \geq j_S(\mu)$. Since $\ell > \ell_\mu$ was arbitrary, we deduce that $\ell_\mu \geq j_S(\mu)$. This concludes the proof of (2.8).

Now let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathscr{M}_+(W_S)$ converging toward $\mu$ in $\mathscr{M}_+(W_S)$. Then for all measurable subset $A \subset D$, we have $\mu_n(A) \to \mu(A)$ and hence, for all $\ell \geq 0$ such that $\mu\{x,\ j_S(x) \geq \ell\} > 0$,

$$\liminf_{n\geq+\infty} \mu_n\{x,\ j_S(x) \geq \ell\} > 0,$$

so that

$$\liminf_{n\geq+\infty} j_S(\mu_n) \geq \ell.$$

This holds true for all $\ell < j_S(\mu)$, so

$$\liminf_{n\geq+\infty} j_S(\mu_n) \geq j_S(\mu),$$

which concludes the proof of the fact that $j_S$ is lower semi-continuous.

*Proof of* (2.9) *and that $j_S(X_n)$ is a.s. non-increasing.* We still assume that $\mu \in \mathscr{M}_+(W_S)$ and that Assumption (A) holds true. Let us first prove that $j_S(\mu) = j_S(\mu S_1)$. We have, for all $\ell \geq 0$,

$$\begin{aligned}\theta_0^{-n} n^{-\ell} (\mu S_1) S_n &= \left(\frac{n}{n+1}\right)^{\ell} \theta_{0,S}\, \theta_{0,S}^{-(n+1)} (n+1)^{-\ell} \mu S_{n+1}\\ &\sim_{n\to+\infty} \theta_{0,S}\, \theta_{0,S}^{-(n+1)} (n+1)^{-\ell} \mu S_{n+1}.\end{aligned}$$

This implies that the lim inf of $\theta_0^{-n} n^{-\ell}(\mu S_1) S_n$ equals 0 if and only if the lim inf of $\theta_{0,S}^{-n} n^{-\ell} \mu S_n$ equals 0, and hence that $j_S(\mu S_1) = j_S(\mu)$.

We conclude by proving the last assertion of the proposition. We have

$$j_S(\mu S_1) = \sup\left\{\ell \geq 0,\ \mu S_1\{x,\ j_S(x) \geq \ell\} > 0\right\},$$

hence

$$\mu S_1\{x,\ j_S(x) > j_S(\mu S_1)\} = 0.$$

Using the equality $j_S(\mu S_1) = j_S(\mu)$ and the fact that $\mu S_1 = \mathbb{P}_\mu(X_1 \in \cdot,\ X_1 \neq \partial)$, we deduce that

$$\mathbb{P}_\mu(j_S(X_1) > j_S(\mu)) = 0,$$

where we used $j_S(\partial) = 0$ due to our notational convention about extension of functions. This and a straightforward application of the Markov property concludes the proof of the proposition. □

We now turn our attention to the implications of Assumption (A) for quasi-stationary distributions. The following proposition considers quasi-stationary distributions $\nu \in \mathcal{M}_+(W_S)$ such that $\nu(\eta_S) > 0$.

**Proposition 2.2.** *Assume that Assumption (A) holds true and let $\nu \in \mathcal{M}_+(W_S)$ be a quasi-stationary distribution such that $\nu(\eta_S) > 0$ and such that $\nu\{j_S(\cdot) \leq \ell\} = 1$ for some $\ell \geq 0$. Then the exponential absorption parameter of $\nu$ is $\theta_{0,S}$.*

*Proof.* According to (2.3), we have for all $x \in D$

$$\theta_{0,S}^{-n} n^{-j_S(x)} \mathbb{E}_x(\mathbb{1}_D(X_n)\mathbb{1}_{n<\tau_\partial}) \xrightarrow[n\to+\infty]{} \eta_S(x), \tag{2.11}$$

and

$$\left|\theta_{0,S}^{-n} n^{-j_S(x)} \mathbb{E}_x(\mathbb{1}_D(X_n)\mathbb{1}_{n<\tau_\partial})\right| \leq |\eta_S(x)| + \alpha_{S,n}|W_S(x)|\,\|\mathbb{1}_D\|_{W_S} \leq C W_S(x). \tag{2.12}$$

Denote by $\theta_\nu$ the absorption parameter of $\nu$. Assume that $\theta_\nu > \theta_{0,S}$, then, for any $\ell \geq 1$ such that $\nu\{j_S(\cdot) \leq \ell\} = 1$, and $n \geq 1$ large enough so that $\theta_\nu^{-n} \leq \theta_{0,S}^{-n} n^{-\ell-1}$,

$$\begin{aligned} \nu(D) &= \theta_\nu^{-n} \mathbb{E}_\nu(\mathbb{1}_D(X_n)\mathbb{1}_{n<\tau_\partial}) \\ &\leq n^{-1} \nu\left(\theta_{0,S}^{-n} n^{-j_S(\cdot)} \mathbb{E}_\cdot(\mathbb{1}_D(X_n)\mathbb{1}_{n<\tau_\partial})\right) \xrightarrow[n\to+\infty]{} 0, \end{aligned}$$

using (2.12). This is a contradiction and hence $\theta_\nu \leq \theta_{0,S}$.

Assume now that $\theta_\nu < \theta_{0,S}$, then Fatou's lemma entails that

$$\begin{aligned}1 = \nu(\mathbb{1}_D) &= \theta_\nu^{-n}\mathbb{E}_\nu(\mathbb{1}_D(X_n)\mathbb{1}_{n<\tau_\partial})\\ &\geq \nu\left(\theta_\nu^{-n} n^{-j_S(\cdot)}\mathbb{E}_\cdot(\mathbb{1}_D(X_n)\mathbb{1}_{n<\tau_\partial})\right) \xrightarrow[n\to+\infty]{} +\infty\end{aligned}$$

by (2.11) and since $\nu(\eta_S) > 0$. Hence, we have proved that $\theta_\nu = \theta_{0,S}$. □

The following proposition shows that all quasi-stationary distributions in $\mathcal{M}_+(W_S)$ with parameter $\theta_{0,S}$ are convex combinations of the $\nu_{S,i}$.

**Proposition 2.3.** *Assume that there exists a QSD $\nu$ with exponential absorption parameter $\theta_{0,S}$. Then*

$$j_S(\nu) = 0 \quad \textit{and} \quad \nu\left\{j_S(\cdot) > 0\right\} = 0.$$

*If in addition Assumption (A) holds true and $\nu \in \mathcal{M}_+(W_S)$, then $\nu(\eta_S) = 1$ and $\nu = \sum_{i\in I_S}\nu(\eta_{S,i})\nu_{S,i}$.*

*Proof.* The property $j_S(\nu) = 0$ is immediate, while the second equality derives immediately from (2.6) in Proposition 2.1.

If in addition Assumption (A) holds true and $\nu \in \mathcal{M}_+(W_S)$, then (2.11) and (2.12) are satisfied. Using the fact that $\nu(W_S) < +\infty$, we deduce from Lebesgue's dominated convergence theorem that

$$\begin{aligned}1 = \nu(\mathbb{1}_D) &= \theta_{0,S}^{-n}\mathbb{E}_\nu(\mathbb{1}_D(X_n)\mathbb{1}_{n<\tau_\partial})\\ &= \nu\left(\theta_{0,S}^{-n} n^{-j_S(\cdot)}\mathbb{E}_\cdot(\mathbb{1}_D(X_n)\mathbb{1}_{n<\tau_\partial})\right) \xrightarrow[n\to+\infty]{} \nu(\eta_S),\end{aligned}$$

which shows that $\nu(\eta_S) = 1$.

Finally, integrating (2.3) with respect to $\nu$ and using the fact that $j_S(x) = 0$ $\nu(\mathrm{d}x)$-almost surely, we deduce that, for all $f \in L^\infty(W_S)$,

$$\left|\theta_{0,S}^{-n}\mathbb{E}_\nu(f(X_n)\mathbb{1}_{n<\tau_\partial}) - \sum_{i\in I_S}\nu(\eta_{S,i})\nu_{S,i}(f)\right| \leq \alpha_{S,n}\nu(W_S)\,\|f\|_{W_S}.$$

Since $\theta_{0,S}^{-n}\mathbb{E}_\nu(f(X_n)\mathbb{1}_{n<\tau_\partial}) = \nu(f)$ and $\nu(W_S) < +\infty$, we deduce that $\nu = \sum_{i\in I_S}\nu(\eta_{S,i})\nu_{S,i}$. This concludes the proof of the proposition. □

In Assumption (A), the $\nu_{S,i}$ do not need to be quasi-stationary distributions. However, we will see in the results of Section 4 that this is typically the case if the set $I_S$ and the measures $\nu_{S,i}$ are defined correctly. In the following corollary, we consider a special situation where this holds true.

**Corollary 2.4.** *Assume that Assumption (A) holds true and that there exists a measurable partition $D = N \cup \left(\bigcup_{i\in I_S} M_i\right)$ such that for all $i \in I_S$, and all $x \in M_i$, we have $\nu_{S,i}(M_i) = 1$, $j_S(x) = 0$, $\eta_{S,i}(x) > 0$, and, for all $i \neq j \in I_S$ and $y \in M_j$, we have $\eta_{S,i}(y) = 0$. Then the quasi-stationary distributions in $\mathscr{M}_+(W_S)$ with absorption parameter $\theta_{0,S}$ are exactly the convex combinations of the probability measures $\nu_{S,i}$. Similarly, the quasi-stationary distributions $\nu$ in $\mathscr{M}_+(W_S)$ such that $\nu(D\setminus N) > 0$ are exactly the convex combinations of the probability measures $\nu_{S,i}$.*

*Remark* 5. Observe that the set $N$ is $\sum_{i\in I_S} \nu_{S,i}$-negligible, but is typically non-empty since it contains all the points with $j_S > 0$ (as we will see in the next sections, $j_S$ is usually non-identically zero in reducible state spaces). △

*Proof.* Proposition 2.3 entails that any quasi-stationary distribution in $\mathscr{M}_+(W_S)$ with absorption parameter $\theta_{0,S}$ is a convex combination of the probability measures $\nu_{S,i}$. Reciprocally, for any $x \in M_i$, we have $\eta_S(x) = \eta_{S,i}(x) > 0$ and hence, according to (2.3),

$$\frac{1}{\eta_S(x)} \sum_{i\in I_S} \eta_{S,i}(x)\nu_{S,i} = \nu_{S,i}$$

is the limit when $n \to +\infty$ of the conditional distribution $\mathbb{P}_x(X_n \in \cdot \mid n < \tau_\partial)$, hence it is a quasi-limiting distribution and thus a quasi-stationary distribution in $\mathscr{M}_+(W_S)$. Moreover $\nu_{S,i}$ satisfies $\nu_{S,i}(M_i) = 1$ and, since $\eta_{S,i} > 0$ on $M_i$, $\nu_{S,i}(\eta_S) > 0$. Proposition 2.2 entails that the absorption parameter of $\nu_{S,i}$ is $\theta_{0,S}$. In particular, for any convex combination $\nu = \sum_{i\in I_S} \lambda_i \nu_{S,i}$, we have $\nu \in \mathscr{M}_+(W_S)$ and

$$\mathbb{P}_\nu(X_n \in \cdot) = \sum_{i\in I_S} \lambda_i \mathbb{P}_{\nu_{S,i}}(X_n \in \cdot) = \sum_{i\in I_S} \lambda_i \theta_{0,S}^n \nu_{S,i} = \theta_{0,S}^n \nu,$$

which implies that $\nu$ is a quasi-stationary distribution with absorption parameter $\theta_{0,S}$.

To conclude the proof, we simply observe that any quasi-stationary distribution $\nu$ in $\mathscr{M}_+(W_S)$ such that $\nu(D \setminus N) > 0$ satisfies $\nu(\eta_S) > 0$ and hence, according to Proposition 2.2, its absorption parameter is $\theta_{0,S}$. □

We conclude this section with properties on the measures $\sum_i \eta_{S,i}(x)\nu_{S,i}$ and on $\eta_S$.

**Proposition 2.5.** *Under Assumption (A):*

*(i)* *For all $x \in D$ such that $\eta_S(x) > 0$, $\frac{1}{\eta_S(x)} \sum_{i\in I_S} \eta_{S,i}(x)\nu_{S,i}$ is a quasi-stationary distribution with absorption parameter $\theta_{0,S}$. In addition, there exists $x \in D$ such that $\eta_S(x) > 0$ and $j_S(x) = 0$.*

*(ii)* *The function $\eta_S$ satisfies, for all all $x \in D$,*

$$\mathbb{E}_x\left[\eta_S(X_n)\mathbb{1}_{j_S(X_n)=j_S(x)}\right] = \theta_{0,S}^n \eta_S(x).$$

*(iii)* *For all $n \geq 0$ and all positive measure $\mu \in \mathscr{M}_+(W_S)$ such that $\mu(\eta_S) > 0$ and $\mu(n^{j_S(\cdot)}W_S) < +\infty$, we have*

$$\left\|\mathbb{P}_\mu\left(X_n \in \cdot \mid n < \tau_\partial\right) - \frac{\sum_{i\in I_S}\mu\left(n^{j_S(\cdot)}\eta_{S,i}\right)\nu_{S,i}}{\sum_{i\in I_S}\mu\left(n^{j_S(\cdot)}\eta_{S,i}\right)}\right\|_{TV} \leq 2\alpha_{S,n}\frac{\mu\left(n^{j_S(\cdot)}W_S\right)}{\mu\left(n^{j_S(\cdot)}\eta_S\right)}. \tag{2.13}$$

*(iv)* *For all measure $\mu \in \mathscr{M}(W_S)$ and all $f \in L^\infty(W_S)$, we have*

$$\left|\theta_{0,S}^{-n} n^{-j_S(|\mu|)}\mathbb{E}_\mu\left(f(X_n)\mathbb{1}_{n<\tau_\partial}\right) - \sum_{i\in I_S}\mu(\mathbb{1}_{j_S(\cdot)=j_S(|\mu|)}\eta_{S,i})\nu_{S,i}(f)\right|$$
$$\leq \left(\alpha_{S,n} + \frac{1}{n}\left\|\sum_{i\in I_S}\eta_{S,i}\nu_{S,i}(W_S)\right\|_{W_S}\mathbb{1}_{j_S(|\mu|)\geq 1}\right)|\mu|(W_S)\,\|f\|_{W_S}, \tag{2.14}$$

*for all $n \geq 1$.*

*Remark* 6. If $j_S$ is not assumed integer valued, then Proposition 2.5 remains true, except that the right hand term in (2.14) has a different form (as will clearly appear in the proof, the term $\frac{1}{n}$ is inherited from the fact that different values taken by $j_S$ are spaced by a distance at least one). △

We start with a preliminary lemma. Under Assumption (A), we have, because of (2.8) in Proposition 2.1, for any $\ell \geq 0$,

$$G_\ell := \left\{\mu \in \mathscr{M}(W_S), j_S(|\mu|) \leq \ell\right\} = \left\{\mu \in \mathscr{M}(W_s), j_S(x) \leq \ell \ |\mu|(\mathrm{d}x)\text{-ae}\right\}.$$

Under Assumption (A), the vector space $G_\ell$, endowed with the norm $\|\cdot\|_{W_S}$, is a Banach space.

**Lemma 2.6.** *Assume that Assumption (A) holds true. For all $\ell \geq 0$, the operator $\mathfrak{S} : G_\ell \to G_\ell$ defined by $\mathfrak{S}\mu = \theta_{0,S}^{-1}\mu S_1$ satisfies Assumption (H) of the Appendix with*

$$\mathfrak{J}_{\mathfrak{S}} = \ell,\ E_{\mathfrak{S}}\mu = \sum_{i\in I_S}\mu\left(\mathbb{1}_{j_S(\cdot)=\mathfrak{J}_{\mathfrak{S}}}\eta_{S,i}\right)\nu_{S,i} \text{ and } \alpha_{\mathfrak{S},n} = \alpha_{S,n} + \frac{1}{n}\left\|\sum_{i\in I_S}\eta_{S,i}\nu_{S,i}(W_S)\right\|_{W_S}\mathbb{1}_{\ell\geq 1}.$$

*Proof of Lemma 2.6.* We have $\sum_{i\in I_S}\eta_{S,i}\nu_{S,i}(W_S)\in L^\infty(W_S)$ by assumption. It follows from (2.3) with $n=1$ that $\mathcal{M}(W_S)$ is stable under $\mathfrak{S}$. Therefore, the stability of $G_\ell$ under $\mathfrak{S}$ is a consequence of (2.9) in Proposition 2.1. Then, for all $\mu\in G_\ell$ and $f\in L^\infty(W_S)$ such that $\|f\|_{W_S}\le 1$, we have, using Assumption (A),

$$\begin{aligned}\left|n^{-\mathfrak{J}_\mathfrak{S}}(\mathfrak{S}^n\mu)(f)-(E_\mathfrak{S}\mu)(f)\right| &= \left|n^{-\ell}\theta_{0,S}^{-n}\mu S_n f-\sum_{i\in I_S}\mu\left(\mathbb{1}_{j_S(\cdot)=\ell}\eta_{S,i}\right)\nu_{S,i}(f)\right|\\ &\le \left|\theta_{0,S}^{-n}\mu\left(n^{-j_S(\cdot)}\mathbb{1}_{j_S(\cdot)=\ell}S_n f\right)-\sum_{i\in I_S}\mu\left(\mathbb{1}_{j_S(\cdot)=\ell}\eta_{S,i}\right)\nu_{S,i}(f)\right|\\ &\quad+\frac{1}{n}\theta_{0,S}^{-n}|\mu|\left(n^{-j_S(\cdot)}\mathbb{1}_{j_S(\cdot)\le\ell-1}S_nW_S\right)\\ &\le \alpha_{S,n}|\mu|\left(\mathbb{1}_{j_S(\cdot)=\ell}W_S\right)+\frac{1}{n}\left[\sum_{i\in I_S}|\mu|\left(\mathbb{1}_{j_S(\cdot)\le\ell-1}\eta_{S,i}\right)\nu_{S,i}(W_S)\right.\\ &\quad\left.+\alpha_{S,n}|\mu|\left(\mathbb{1}_{j_S(\cdot)\le\ell-1}W_S\right)\right].\end{aligned}$$

This implies that $\|n^{-\mathfrak{J}_\mathfrak{S}}\mathfrak{S}^n\mu-E_\mathfrak{S}\mu\|_{W_S}\to 0$ when $n\to+\infty$. Since $G_\ell$ is a closed subset of the Banach space $\mathcal{M}(W_S)$, we deduce that $E_\mathfrak{S}\mu\in G_\ell$ for all $\mu\in G_\ell$. Hence (H) is proved. □

*Proof. Proof of (i).* Fix $x_*\in D$ such that $\eta_S(x_*)>0$ and let $\ell_*=j_S(x_*)$. It follows from Lemma 2.6 that $\mathfrak{S}\mu=\theta_{0,S}^{-1}\mu S_1$ satisfies (H) on the Banach space $G_{\ell_*}$. Using Proposition A.1 of Appendix A.1, we deduce that the operator

$$E_\mathfrak{S}\mu=\sum_{i\in I_S}\mu\left(\mathbb{1}_{j_S(\cdot)=\mathfrak{J}_\mathfrak{S}}\eta_{S,i}\right)\nu_{S,i}$$

on $G_{\ell_*}$, with $\mathfrak{J}_\mathfrak{S}=\ell_*$, satisfies

$$(E_\mathfrak{S}\mathfrak{S})\delta_x=(\mathfrak{S}E_\mathfrak{S})\delta_x=E_\mathfrak{S}\delta_x,\ \text{for all } x\in D \text{ such that } j_S(x)\le\ell_*.$$

This means that, for all $x\in D$ such that $j_S(x)\le\ell_*$,

$$\theta_{0,S}^{-1}\sum_{i\in I_S}\delta_xS_1\left(\mathbb{1}_{j_S(\cdot)=\mathfrak{J}_\mathfrak{S}}\eta_{S,i}\right)\nu_{S,i}=\mathbb{1}_{j_S(x)=\ell_*}\theta_{0,S}^{-1}\sum_{i\in I_S}\eta_{S,i}(x)\nu_{S,i}S_1 \tag{2.15}$$

$$=\mathbb{1}_{j_S(x)=\ell_*}\sum_{i\in I_S}\eta_{S,i}(x)\nu_{S,i}. \tag{2.16}$$

Since $j_S(x_*)=\ell_*$, we deduce from the second equality that $\nu:=\frac{1}{\eta_S(x_*)}\sum_{i\in I_S}\eta_{S,i}(x_*)\nu_{S,i}\in\mathcal{M}_+(W_S)$ is a quasi-stationary distribution for $S_1$ with absorption parameter

$\theta_{0,S}$. In addition, according to Proposition 2.3, we have $\nu(\eta_S) = 1$ and $\nu(\mathbb{1}_{j_S(\cdot)=0}) = 1$, so that $\nu(\mathbb{1}_{j_S(\cdot)=0}\eta_S) = 1$. Therefore, there exists $x \in D$ such that $\eta_S(x) > 0$ and $j_S(x) = 0$.

*Proof of (ii).* Fix $x \in D$. Then, applying as above Proposition A.1 of Appendix A.1 on the Banach space $G_{j_S(x)}$, we obtain (2.15) and (2.16) with $\ell_*$ replaced by $j_S(x)$. Integrating on both sides the test function $\mathbb{1}_D$, this implies that

$$\eta_S(x) = \sum_{i\in I_S} \eta_{S,i}(x) = \theta_{0,S}^{-1} \sum_{i\in I_S} \delta_x S_1\left(\mathbb{1}_{j_S(\cdot)=j_S(x)}\eta_{S,i}\right) = \theta_{0,S}^{-1}\delta_x S_1\left(\mathbb{1}_{j_S(\cdot)=j_S(x)}\eta_S\right).$$

Hence

$$\mathbb{E}_x\left[\eta_S(X_1)\mathbb{1}_{j_S(X_1)=j_S(x)}\right] = \theta_{0,S}\eta_S(x).$$

We deduce that, for all $n \geq 1$, $\mathbb{P}_x$-almost surely,

$$\mathbb{E}_{X_{n-1}}\left[\eta_S(X_1)\mathbb{1}_{j_S(X_1)=j_S(x)}\right]\mathbb{1}_{j_S(X_{n-1})=j_S(x)} = \theta_{0,S}\eta_S(X_{n-1})\mathbb{1}_{j_S(X_{n-1})=j_S(x)}.$$

Taking the expectation and using the Markov property, we deduce that

$$\mathbb{E}_x\left[\eta_S(X_n)\mathbb{1}_{j_S(X_n)=j_S(x)}\mathbb{1}_{j_S(X_{n-1})=j_S(x)}\right] = \theta_{0,S}\mathbb{E}_x\left[\eta_S(X_{n-1})\mathbb{1}_{j_S(X_{n-1})=j_S(x)}\right].$$

Because of the last assertion of Proposition 2.1, we deduce that $\{j_S(X_n) = j_S(x)\} = \{j_S(X_n) = j_S(X_{n-1}) = \cdots = j_S(x)\}$ up to a $\mathbb{P}_x$-negligible event, so that

$$\mathbb{E}_x\left[\eta_S(X_n)\mathbb{1}_{j_S(X_n)=j_S(x)}\right] = \theta_{0,S}\mathbb{E}_x\left[\eta_S(X_{n-1})\mathbb{1}_{j_S(X_{n-1})=j_S(x)}\right].$$

Then the property (ii) follows by induction.

*Proof of (iii).* Fix $n \geq 1$ and let $\mu \in \mathcal{M}_+(W_S)$ such that $\mu(\eta_S) > 0$ and $\mu(n^{j_S(\cdot)}W_S) < +\infty$. Integrating (2.3) with respect to the measure $n^{j_S(x)}\mu(\mathrm{d}x)$ we obtain, for all $f \in L^\infty(W_S)$ such that $\|f\|_{W_S} \leq 1$,

$$\left|\theta_{0,S}^{-n}\mathbb{E}_\mu(f(X_n)\mathbb{1}_{n<\tau_\partial}) - \sum_{i\in I_S}\mu\left(n^{j_S(\cdot)}\eta_{S,i}\right)\nu_{S,i}(f)\right| \leq \alpha_{S,n}\,\mu(n^{j_S(\cdot)}W_S). \tag{2.17}$$

Then we have for all $f$ measurable bounded by 1

$$\begin{aligned}\left|\frac{\mathbb{E}_\mu f(X_n)\mathbb{1}_{n<\tau_\partial}}{\mathbb{P}_\mu(n<\tau_\partial)}-\frac{\sum_{i\in I_S}\mu\left(n^{j_S(\cdot)}\eta_{S,i}\right)\nu_{S,i}(f)}{\sum_{i\in I_S}\mu\left(n^{j_S(\cdot)}\eta_{S,i}\right)}\right| \\ &\leq \frac{\mathbb{E}_\mu f(X_n)\mathbb{1}_{n<\tau_\partial}}{\mathbb{P}_\mu(n<\tau_\partial)}\frac{|\sum_{i\in I_S}\mu\left(n^{j_S(\cdot)}\eta_{S,i}\right)-\theta_{0,S}^{-n}\mathbb{P}_\mu(n<\tau_\partial)|}{\sum_{i\in I_S}\mu\left(n^{j_S(\cdot)}\eta_{S,i}\right)} \\ &\qquad+\frac{|\theta_{0,S}^{-n}\mathbb{E}_\mu f(X_n)\mathbb{1}_{n<\tau_\partial}-\sum_{i\in I_S}\mu\left(n^{j_S(\cdot)}\eta_{S,i}\right)\nu_{S,i}(f)|}{\sum_{i\in I_S}\mu\left(n^{j_S(\cdot)}\eta_{S,i}\right)} \\ &\leq 2\alpha_{S,n}\frac{\mu\left(n^{j_S(\cdot)}W_S\right)}{\mu\left(n^{j_S(\cdot)}\eta_S\right)},\end{aligned}$$

which concludes the proof.

*Proof of (iv).* Let $\mu$ be a measure in $\mathcal{M}(W_S)$. The property (iv) is an immediate consequence of Lemma 2.6 with $\ell = j_S(|\mu|)$. □

*Remark* 7. A natural alternative to Assumption (A) is the following, where $j_S$ is replaced by an arbitrary function.

**Assumption (A').** We have $\theta_{0,S}\in(0,1]$, there exists a measurable, bounded, integer-valued function $j:D\to\mathbb{Z}_+$, a measurable function $W_S:D\to[1,+\infty)$, a finite or countable set $I_S$, some probability measures $\nu_{S,i}\in\mathcal{M}(W_S)$ and non-identically zero non-negative $\eta_{S,i}\in L^\infty(W_S)$ for each $i\in I_S$, such that

$$\sum_{i\in I_S}\eta_{S,i}\nu_{S,i}(W_S)\in L^\infty(W_S) \tag{2.18}$$

and such that, for all $f\in L^\infty(W_S)$, all $n\geq 1$ and all $x\in D$,

$$\left|\theta_{0,S}^{-n}n^{-j(x)}\mathbb{E}_x(f(X_n)\mathbb{1}_{n<\tau_\partial})-\sum_{i\in I_S}\eta_{S,i}(x)\nu_{S,i}(f)\right|\leq\alpha_{S,n}W_S(x)\|f\|_{W_S}, \tag{2.19}$$

where $\alpha_{S,n}$ goes to 0 when $n\to+\infty$.

It is clear that, if (A') is satisfies, then $j_S\leq j$ and, if $x\in D$ is such that $\eta_S(x)>0$, then $j_S(x)=j(x)$. In particular, Assumption (A') is less demanding than Assumption (A) and may be deduced under less stringent conditions (see next section). However, $j$ lacks most of the properties proved for $j_S$ above and, in order to adapt the proof of most of our results, one needs in addition to impose the following condition on $j$:

**Assumption (A").** Assumption (A') holds true and, in addition, $(j(X_n))_{n\in\mathbb{N}}$ is $\mathbb{P}_x$-almost surely decreasing for all $x \in D$.

Since this last property is verified by $j_S$, as proved in Proposition 2.1, Assumption (A) implies Assumption (A"). Most of the results of the next sections can be extended using Assumpion (A") instead of (A). However, it seems difficult to find situations where Assumption (A") is satisfied but Assumption (A) fails, and the sufficient criteria for Assumption (A) obtained in the next section allow the explicit computation of $j_S$, which would of course not be the case for a similar study with Assumption (A") instead. This is why we mainly focus on Assumption (A) in this paper.

To conclude this remark, we comment on the extension of the results of this section under the weaker Assumption (A') or (A"). Proposition 2.2 and its proof holds true under the weaker Assumption (A'), for quasi-stationary distributions $\nu$ such that $\nu\{j(\cdot) \leq \ell\} = 1$ for some $\ell \geq 0$. Concerning Proposition 2.3, under the weaker Assumption (A'), we have $\nu\left\{j(\cdot) > 0 \text{ and } \eta_S(\cdot) > 0\right\} = 0$, and, under Assumptions (A') or (A"), one may have $\nu(\eta_S) = 0$. Corollary 2.4 does not hold true under Assumption (A') or (A"), however, under Assumption (A'), for any $x \in D$ such that $\mu := \sum_{i\in I_S} \eta_{S,i}(x)\nu_{S,i} \neq 0$, the probability measure $\mu/\mu(E)$ is a quasi-limiting distribution, and thus a quasi-stationary distribution. Finally, Proposition 2.5 remains partially true under the weaker Assumption (A"), replacing $j_S$ by $j$ in the statements: the first part of (i) remains true, the second part of (i) does not stand true, (ii) and (iii) remain true, and (iv) holds true with $j_S(|\mu|)$ replaced by $j(|\mu|) := \inf\{\ell \geq 0,\ j(x) \leq \ell\ |\mu|(\mathrm{d}x)\text{-ae}\}$. △

# 3 Quasi-stationary distributions in reducible state spaces with two successive sets

We start our study of the quasi-stationary distribution for reducible processes by focusing on cases where the state space can be separated into two successive classes. This is the generic situation that can be used iteratively to treat more complicated cases (see Section 4).

We consider a discrete time Markov process $(X_n, n \in \mathbb{Z}_+)$ evolving in a measurable set $D \cup \{\partial\}$ with absorption at $\partial \notin D$ at time $\tau_\partial$, and sub-Markovian semigroup $(S_n)_{n\in\mathbb{Z}_+}$. We assume that the transition probabilities of $X$ satisfy the structure displayed in Figure 1: there is a partition $\{D_1, D_2\}$ of $D$ such that the process starting from $D_1$ can access $D_1 \cup D_2 \cup \{\partial\}$ and the process starting from

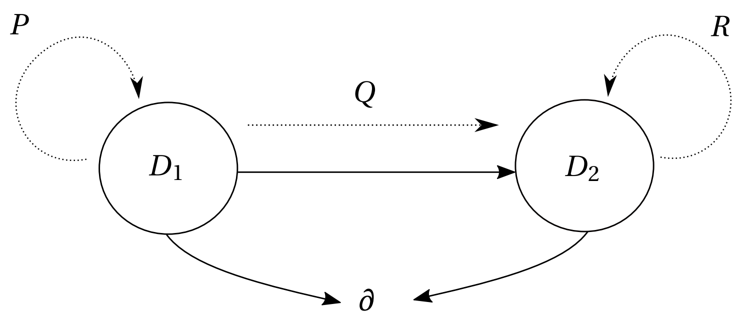


Figure 1: Transition graph displaying the relations between the sets $D_1$, $D_2$ and $\partial$. The dashed lines indicate the domains and co-domains of the sub-Markov kernels $P, Q, R$.

$D_2$ can only access $D_2 \cup \{\partial\}$. More formally, we assume that $\mathbb{P}_x(T_{D_1} = +\infty) = 1$ for all $x \in D_2$, where we denote, for any measurable set $A \subset D$, $T_A = \inf\{n \in \mathbb{Z}_+,\ X_n \in A\}$.

We denote by $(P_n)$ the sub-Markovian semigroup of the process $X$ restricted to $D_1$, by $(R_n)$ the sub-Markovian semigroup of the processes $X$ restricted to $D_2$ and by $Q$ the transition kernel from $D_1$ to $D_2$ for $X$. More formally, for all measurable $f : D_1 \to [0, +\infty)$ and $g : D_2 \to [0, +\infty)$, for all $x \in D_1$ and $y \in D_2$, we define

$$P_n f(x) = \mathbb{E}_x(f(X_n)),\ R_n g(y) = \mathbb{E}_y(g(X_n)) \text{ and } Qg(x) = \mathbb{E}_x(g(X_1)).$$

Note that, due to our notational convention about extensions of functions by 0 outside of their domain, the previous definitions mean

$$P_n f(x) = \mathbb{E}_x(f(X_n)\mathbb{1}_{n<T_{D_2\cup\partial}}),\ R_n g(y) = \mathbb{E}_y(g(X_n)\mathbb{1}_{n<\tau_\partial}) \text{ and } Qg(x) = \mathbb{E}_x(g(X_1)\mathbb{1}_{X_1\in D_2}).$$

We consider now three different situations (which are in correspondence with the three cases studied in Appendix A). The constants $\theta_{0,P}$ and $\theta_{0,R}$ denote respectively the exponential convergence parameters of the semigroups $(P_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$.

**Assumption (A1)** We have $j_{0,P} < +\infty$, the process $X$ restricted to $D_1$ satisfies Assumption (A) and there exists a measurable function $W_R : D_2 \to [1, +\infty)$ such that, for some constants $\gamma \in [0, \theta_{0,P})$ and $c_1 > 0$, for all $x \in D_1$ and $y \in D_2$,

$$\mathbb{E}_x(W_R(X_1)) \leq W_P(x) \text{ and } \mathbb{E}_y(W_R(X_n)) \leq c_1 \gamma^n W_R(y),\ \forall n \geq 0. \tag{3.1}$$

**Assumption (A2)** We have $j_{0,R} < +\infty$, the process $X$ restricted to $D_2$ satisfies Assumption (A) and there exists a measurable function $W_P : D_1 \to [1, +\infty)$ such that, for some constants $\gamma \in [0, \theta_{0,R})$ and $c_2 > 0$, for all $x \in D_1$,

$$\mathbb{E}_x(W_R(X_1)) \le W_P(x) \text{ and } \mathbb{E}_x(W_P(X_n)) \le c_2 \gamma^n W_P(x),\ \forall n \ge 0.$$

**Assumption (A3)** We have $j_{0,P} = 0$, $j_{0,R} < +\infty$ and $\theta_{0,R} = \theta_{0,P}$. In addition, the process $X$ restricted to $D_1$ satisfies Assumption (A) with $\eta_P > 0$, and the process $X$ restricted to $D_2$ also satisfies Assumption (A). Finally,

$$\mathbb{E}_x(W_R(X_1)) \le W_P(x),\ \forall x \in D_1, \tag{3.2}$$

and there exists $\ell_* \in \mathbb{Z}_+$ such that, for all $x \in D_1$ and all $i \in I_P$,

$$\mathbb{P}_x\left(j_R(X_{T_{D_2}}) \le \ell_* \text{ and } T_{D_2} < +\infty\right) = \mathbb{P}_x(T_{D_2} < +\infty) \tag{3.3}$$

$$\text{and} \quad \mathbb{P}_{\nu_{P,i}}\left(j_R(X_{T_{D_2}}) = \ell_* \text{ and } \eta_R(X_{T_{D_2}}) > 0 \text{ and } T_{D_2} < +\infty\right) > 0, \tag{3.4}$$

where we recall that $\eta_R = \sum_{k \in I_R} \eta_{R,k}$. Note that $\ell_* \le j_{0,R}$.

*Remark* 8. Note that Assumption (A) remains valid if the function $W_S$ is multiplied by a positive constant. Hence, in the above assumptions (A1), (A2) and (A3), the requirement $\mathbb{E}_x(W_R(X_1)) \le W_P(x)$ is actually equivalent to $\mathbb{E}_x(W_R(X_1)) \le C\, W_P(x)$ for some positive constant $C > 0$. △

*Remark* 9. In (A3), the assumptions $j_{0,P} = 0$ and the fact that (3.3) is satisfied for all $x \in D_1$ may seem restrictive conditions. However, we will see in Section 4 that, applying this property inductively in a precise order, this is sufficient to obtain a convergence result similar to (2.3) with non-zero $j_{0,S}$ (in cases with a finite or denumerable number of communication classes under mild assumptions). △

*Remark* 10. Possible candidates for $W_R \ge 1$ in Assumption (A1) (resp. $W_P \ge 1$ in Assumption (A2)) are the exponential moment of exit times from $D_2$ (resp. from $D_1$). Indeed, if $W_R(y) = \mathbb{E}_y(\gamma^{-\tau_\partial})$ is finite for all $y \in D_2$, then $\mathbb{E}_y(W_R(X_1)) \le \gamma W_R(y)$ for all $y \in D_2$. Similarly, if $W_P(x) = \mathbb{E}_x\left(\gamma^{-T_{D_2 \cup \{\partial\}}}\right)$ is finite for all $x \in D_1$, then $\mathbb{E}_x(W_P(X_1)) \le \gamma W_P(x)$ for all $x \in D_1$. Proofs of these classical results can be found e.g. in [24]. △

Under each of the above assumptions, we prove that Assumption (A) actually holds true for the process $X$ evolving on $D = D_1 \cup D_2$ before absorption, with formal expressions of the parameters $W_S$, $j_S$, $\nu_S$ and $\alpha_{S,n}$. In the following result, we set $\alpha_{P,0} = \alpha_{R,0} = 1$.

**Theorem 3.1.** *Assume that either Assumption (A1), or (A2), or (A3) holds true. Then $X$ satisfies Assumption (A) with $W_S = W_P + W_R$. Moreover, there exists a constant $C > 0$, independent of $x \in D$ and $n \in \mathbb{N}$, such that*

*(i) under Assumption (A1), we have $\theta_{0,S} = \theta_{0,P}$, $j_S = j_P$, and, for all $i \in I_S = I_P$, $\eta_{S,i} \propto \eta_{P,i}$ and*

$$\nu_{S,i} \propto \nu_{P,i} + \sum_{k\geq 0} \theta_{0,S}^{-k-1} \mathbb{P}_{\nu_{P,i}} \left( T_{D_2} = 1,\ X_{k+1} \in \cdot \right),$$

*with inverse proportionality constants, and*

$$\alpha_{S,n} = C \sum_{k=0}^{n} \left( \frac{\gamma}{\theta_{0,P}} \right)^k \left( (1 + j_{0,P}) \alpha_{P,n-k} + j_{0,P} \frac{k}{n} \right);$$

*(ii) under Assumption (A2), we have $\theta_{0,S} = \theta_{0,R}$ and, for all $x \in D$,*

$$j_S(x) = \begin{cases} \max_{n\geq 0} j_R(\delta_x P^n Q) & \text{if } x \in D_1, \\ j_R(x) & \text{if } x \in D_2 \end{cases}$$

*and for all $i \in I_S = I_R$,*

$$\eta_{S,i}(x) = \mathbb{E}_x \left( \theta_{0,R}^{-T_{D_2}} \eta_{R,i}(X_{T_{D_2}}) \mathbb{1}_{j_R(X_{T_{D_2}}) = j_S(x)} \right)$$

*$\nu_{S,i} = \nu_{R,i}$ and*

$$\alpha_{S,n} = C \sum_{k=0}^{n} \left( \alpha_{R,n-k} + j_{0,R} \frac{k}{n} \right) \left( \frac{\gamma}{\theta_{0,R}} \right)^k;$$

*(iii) under Assumption (A3), we have $\theta_{0,S} = \theta_{0,R} = \theta_{0,P}$,*

$$j_S(x) = \begin{cases} 1 + \ell_* & \text{for all } x \in D_1, \\ j_R(x) & \text{for all } x \in D_2, \end{cases}$$

*where $\ell_*$ is defined in* (3.3)–(3.4), *$I_S = I_R$, for all $i \in I_R$, $\nu_{S,i} = \nu_{R,i}$,*

$$\eta_{S,i}(x) = \eta_{R,i}(x) + \frac{\theta_{0,P}^{-1}}{1 + \ell_*} \sum_{k\in I_P} \eta_{P,k}(x) \mathbb{E}_{\nu_{P,k}} \left( \eta_{R,i}(X_1) \mathbb{1}_{j_R(X_1) = \ell_*} \right), \quad \forall x \in D,$$

*and*

$$\alpha_{S,n} = \frac{C}{n} \left( \ell_* + \sum_{k=0}^{n} \left( \alpha_{P,k} + \alpha_{R,k} \frac{k^{\ell_*}}{n^{\ell_*}} \right) \right),$$

*Remark* 11. Note that, in conclusion (iii) of the last theorem, even if $\alpha_{P,n}$ and $\alpha_{S,n}$ converge geometrically to 0, $\alpha_{S,n}$ only converges to 0 in $O(1/n)$. Similarly, in conclusion (i) (resp. (ii)), if $j_P$ (resp. $j_R$) is not identically equal to 0, then the convergence rate is $O(1/n)$. △

*Remark* 12. Theorem 3.1 under Assumption (A1) and its proof adapt to the situation where $j_P$ takes finite values but where $j_{0,P} = +\infty$, in which case $\alpha_{S,n}$ depends on the initial position $x$. Similarly, under Assumption (A2) or (A3), the result adapts to the situation where $j_R$ takes finite values but where $j_{0,R} = +\infty$ (and with the additional assumption that $\max_{n\geq 0} j_R(\delta_x P^n Q) < +\infty$ for all $x \in D_1$ when working under Assumption (A2)), in which case $\alpha_{S,n}$ also depends on $x$. Thus, in these situations, one checks that $S$ satisfies a version of Assumption (A) allowing dependence of $\alpha_{S,n}$ with respect to $x$.

In fact, it is possible to adapt the results of this paper to the situation where $\alpha_{S,n}$ (and hence $\alpha_{R,n}$ and/or $\alpha_{P,n}$) is allowed to depend on $x$. However, this leads to additional technical assumptions and cumbersome expressions. Since the case $\alpha$ independent of $x$ is already rich enough and is sufficiently powerful to derive general results (see Sections 4 and 5), we decided to focus on this situation and leave the adaptation to the interested reader (see also Remark 23 in Appendix A.2 for the expressions obtained in this situation).

Finally, if $j_S$ is not assumed integer valued, then again Theorem 3.1 holds true, but with $\alpha_{S,n}$ depending non-explictly on the initial position of the process. △

We will see in the proof that a weakened version of Assumption (A3), without the requirement $\eta_P > 0$ in $D_1$ and without (3.4) implies Assumption (A") (as introduced in Remark 7). This entails the following corollary.

**Corollary 3.2.** *Assume that $j_{0,P} = 0$, $j_{0,R} < +\infty$, $\theta_{0,R} = \theta_{0,P}$, the process $X$ restricted to $D_1$ satisfies Assumption (A) and the process $X$ restricted to $D_2$ also satisfies Assumption (A). Assume also that* (3.2) *holds true and define $\ell_* \in \mathbb{Z}_+$ as the smallest $\ell$ such that, for all $x \in D_1$, $\mathbb{P}_x(j_R(X_{T_{D_2}}) \leq \ell) = 1$. Then for all $f \in L^\infty(W_S)$, all $n \geq 1$ and all $x \in D$,*

$$\Bigg|\theta_{0,P}^{-n} n^{-j(x)} \mathbb{E}_x(f(X_n)\mathbb{1}_{n<\tau_\partial}) - \sum_{i\in I_R}\left(\eta_{R,i}(x) + \frac{\theta_{0,P}^{-1}}{1+\ell_*}\sum_{k\in I_P}\eta_{P,k}(x)\mathbb{E}_{\nu_{P,k}}\left(\mathbb{1}_{j_R(X_1)=\ell_*}\eta_{R,i}(X_1)\right)\right)\nu_{R,i}(f)\Bigg| \leq \alpha_{S,n} W_S(x)\|f\|_{W_S}, \quad (3.5)$$

*where*

$$j(x) = \begin{cases} 1+\ell_* & \text{for all } x \in D_1, \\ j_R(x) & \text{for all } x\in D_2, \end{cases}$$

*and*

$$\alpha_{S,n} = \frac{C}{n}\left(\ell_* + \sum_{k=0}^{n}(\alpha_{P,k}+\alpha_{R,k})\right),$$

*for some constant $C>0$, which does not depend on $x$ and $n$. In addition, $j_R(x) = j(x)$ for all $x \in D_2$ and $j_R(x) \leq j(x)$ for all $x \in D_1$.*

*Remark* 13. The difference between Theorem 3.1 (iii) and Corollary 3.2 is that we don't know if the polynomial exponent $j$ involved in (3.5) is equal on $D_1$ to the polynomial convergence parameter $j_S$ defined in (2.1). However, it follows from (3.5) that $j(x) = j_S(x)$ for all $x \in D_1$ such that

$$\sum_{i\in I_R}\sum_{k\in I_P}\eta_{P,k}(x)\mathbb{E}_{\nu_{P,k}}\left(\mathbb{1}_{j_R(X_1)=\ell_*}\eta_{R,i}(X_1)\right) > 0.$$

The introduction of an arbitrary polynomial exponent $j$ instead $j_S$ is the subject of Assumptions (A') and (A") introduced in Remark 7. Using very similar proofs, processes with two successive sets can also be studied using Assumption (A") in place of Assumption (A), then loosing the explicit expression of the polynomial convergence parameter. △

*Proof of Theorem 3.1 and Corollary 3.2.* Under Assumptions (A1) or (A2) or (A3), we define the linear operators $\mathfrak{P} : \mathcal{M}(W_P) \to \mathcal{M}(W_P)$, $\mathfrak{Q} : \mathcal{M}(W_P) \to \mathcal{M}(W_R)$ and $\mathfrak{R} : \mathcal{M}(W_R) \to \mathcal{M}(W_R)$ (these notations implicitely assume that $\mathcal{M}(W_P) \subset \mathcal{M}(D_1)$ and $\mathcal{M}(W_R) \subset \mathcal{M}(D_2)$) by

$$\mathfrak{P}\mu = \theta_0^{-1}\mu P_1, \quad \mathfrak{Q}\mu = \theta_0^{-1}\mu Q, \quad \text{and } \mathfrak{R}\mu = \theta_0^{-1}\mu R_1,$$

where $\theta_0 = \theta_{0,P}$ under Assumption (A1), $\theta_0 = \theta_{0,R}$ under Assumption (A2), and $\theta_0 = \theta_{0,P} = \theta_{0,R}$ under Assumption (A3). Under Assumption (A1), (A2) or (A3), all these operators are bounded. Our aim is to apply the results of Appendix A.2 to $\mathfrak{S} : \mathcal{M}(W_S) \to \mathcal{M}(W_S)$, where $\mathcal{M}(W_S) \equiv \mathcal{M}(W_P) \oplus \mathcal{M}(W_R)$, with $W_S = W_R + W_P$ and $\mathfrak{S} = \mathfrak{P} + \mathfrak{Q} + \mathfrak{R}$. Beware that $\mathfrak{P}, \mathfrak{R}, \mathfrak{Q}, \mathfrak{S}$ act on the left on $\mu$ while $P_n, R_n, Q, S_n$ act on the right, so that, for instance, $\mathfrak{R}\mathfrak{P}\mu = \theta_0^{-2}\mu P_1 R_1$.

**Under Assumption (A1).** We make use of Proposition A.2. We define $B_2$ as the Banach space $\mathcal{M}(W_R)$ and observe that the operator $\mathfrak{R} : B_2 \to B_2$ is bounded and

$$\sum_{n=0}^{\infty} |||\mathfrak{R}^n||| \le \sum_{n=0}^{\infty} \theta_{0,P}^{-n} \sup_{\mu\in B_2,\ |\mu|(W_R)=1} |\mu| R_n W_R.$$

It follows from (3.1) that

$$\sum_{n=0}^{\infty} |||\mathfrak{R}^n||| \le c_1 \sum_{n=0}^{\infty} \theta_{0,P}^{-n}\gamma^n \sup_{\mu\in B_2,\ |\mu|(W_R)=1} |\mu|(W_R) = \frac{c_1\theta_0}{\theta_{0,P}-\gamma}.$$

Moreover, $\Gamma_n \le c_1 \frac{\gamma^n/\theta_{0,P}^{n-1}}{\theta_{0,P}-\gamma}$ and $\gamma_n \le c_1\gamma^n/\theta_{0,P}^n$ (using the notations of Proposition A.2). In particular, if $x \in D_2$, then (2.3) holds true with $\eta_{S,i}(x) = 0$ for all $i \in I_S = I_P$, $j_S(x) = 0$, $W_S(x) = W_R(x)$ and $\alpha_{S,n} = c_1(\frac{\gamma}{\theta_{0,P}})^n$.

From now on we assume that $x \in D_1$ and consider the vector space $B_1 = \{\mu \in \mathcal{M}(W_P),\ j_P(|\mu|) \le j_P(x)\}$. By Lemma 2.6, the operator $\mathfrak{P} : B_1 \to B_1$ satisfies Assumption (H) form the Appendix with $\mathfrak{J}_{\mathfrak{P}} = j_P(x)$,

$$E_{\mathfrak{P}}\mu = \sum_{i\in I_P} \mu(\mathbb{1}_{j_P(\cdot)=j_P(x)}\eta_{P,i})\nu_{P,i} \tag{3.6}$$

and

$$\alpha_{\mathfrak{P},n} = \alpha_{P,n} + \mathbb{1}_{j_P(x)\ge 1} \frac{\left\|\sum_{i\in I_P}\eta_{P,i}\nu_{P,i}(W_P)\right\|_{W_P}}{n}$$

Note also that $\mathfrak{Q} : B_1 \to B_2$ is a bounded operator by (3.1). As a consequence, according to Proposition A.2, $\mathfrak{S}$ restricted to $B = B_1 \oplus B_2$ also satisfies Assumption (H) with $\mathfrak{J}_{\mathfrak{S}} = j_P(x)$ and for all $\mu \in B_1$,

$$\begin{aligned}E_{\mathfrak{S}}\mu &= E_{\mathfrak{P}}\mu + \sum_{k\ge 0}\mathfrak{R}^k\mathfrak{Q}E_{\mathfrak{P}}\mu = \sum_{i\in I_P}\mu(\eta_{P,i})\nu_{P,i} + \sum_{k\ge 0}\theta_{0,P}^{-k-1}\sum_{i\in I_P}\mu(\eta_{P,i})\mathbb{P}_{\nu_{P,i}}\left(T_{D_2}=1,\ X_{k+1}\in\cdot\right)\\ &= \sum_{i\in I_P}\mu(\eta_{P,i})\left(\nu_{P,i} + \sum_{k\ge 0}\theta_{0,P}^{-k-1}\mathbb{P}_{\nu_{P,i}}\left(T_{D_2}=1,\ X_{k+1}\in\cdot\right)\right).\end{aligned}$$

and

$$\begin{aligned}\alpha_{\mathfrak{S},n} &= \alpha_{\mathfrak{P},n} + C\Gamma_n + C\sum_{k=0}^{n-1}\gamma_k\left(\frac{(\alpha_{\mathfrak{P},n-k-1}+1)\mathfrak{J}_{\mathfrak{P}}(k+1)}{n} + \alpha_{\mathfrak{P},n-k-1}\right)\\ &\le C\sum_{k=0}^{n}\left(\frac{\gamma}{\theta_{0,P}}\right)^k\left((1+j_P(x))\alpha_{P,n-k} + \mathbb{1}_{j_P(x)\ge 1}\frac{1}{n+1-k} + j_P(x)\frac{k}{n}\right) \qquad (3.7)\\ &\le \alpha_{S,n} := C\sum_{k=0}^{n}\left(\frac{\gamma}{\theta_{0,P}}\right)^k\left((1+j_{0,P})\alpha_{P,n-k} + j_{0,P}\frac{k}{n}\right)\end{aligned}$$

for some constant $C > 0$ that may change from line to line, where we used $\mathbb{1}_{j_{0,P}\geq 1} \leq j_{0,P}$ and $\frac{1}{n-k+1} \leq \frac{k}{n}$. Using the fact that $\mathfrak{S}^n\mu = \theta_{0,P}^{-n}\mu S_n$ and taking $\mu = \delta_x$, we deduce that, for all $x \in D$ and all $f \in L^\infty(W_S)$,

$$\left| n^{-j_P(x)}\theta_{0,P}^{-n} S_n f(x) - \sum_{i\in I_P} \eta_{P,i}(x)\left(\nu_{P,i}(f) + \sum_{k\geq 0}\theta_{0,P}^{-k-1}\mathbb{E}_{\nu_{P,i}}\left(\mathbb{1}_{T_{D_2}=1} f(X_{k+1})\right)\right)\right| \\ \leq \alpha_{S,n} W_S(x)|f|. \quad (3.8)$$

It only remains to prove that $j_S(x) = j_P(x)$ for all $x \in D$ (recall that under our convention $j_P$ is extended to $D_2$ by the value 0). On the one hand, the definitions of $j_S$, $j_P$ and $S$ clearly imply that $j_S(x) \geq j_P(x)$ for all $x \in D$. On the other hand, inequality (3.8) implies that, for all $\varepsilon > 0$,

$$\liminf_{n\to+\infty} n^{-(j_P(x)+\varepsilon)}\theta_{0,P}^{-n} S_n \mathbb{1}_D(x) = 0,$$

so that $j_S(x) \leq j_P(x) + \varepsilon$ for all $\varepsilon > 0$, and hence $j_S(x) \leq j_P(x)$. This concludes the proof of (i).

**Under Assumption (A2).** We make use of Proposition A.3. For all $x \in D_2$, we have $\delta_x S_n = \delta_x R_n$, so that (2.3) holds true with $I_S = I_R$, $\eta_{S,i}(x) = \eta_{R,i}(x)$, $j_S(x) = j_R(x)$ and $\alpha_{S,n} = \alpha_{R,n}$. This also implies that $\theta_S(x) = \theta_R(x)$ for all $x \in D_2$.

We fix now $x \in D_1$. We set

$$\mathfrak{J}(x) := \max_{n\geq 0} j_R(\delta_x P^n Q)$$

and consider the operators $\mathfrak{P}$, $\mathfrak{R}$ and $\mathfrak{S}$ restricted to the Banach space

$$B = B_1 \oplus B_2 \subset \mathcal{M}(W_S),$$

where

$$B_1 = \left\{\mu \in \mathcal{M}(W_P),\ \max_{n\geq 0} j_R(|\mu|P^n Q) \leq \mathfrak{J}(x)\right\} \text{ and } B_2 = \left\{\mu \in \mathcal{M}(W_R),\ j_R(|\mu|) \leq \mathfrak{J}(x)\right\}.$$

Note that $B$ is indeed stable by $\mathfrak{P}$, $\mathfrak{R}$ and $\mathfrak{S}$. In addition, Proposition 2.1 entails that $B_1$ is a Banach subspace of $\mathcal{M}(W_P)$.

We first observe that

$$\sum_{n=0}^{\infty} |||\mathfrak{P}^n||| \leq \sum_{n=0}^{\infty} \theta_{0,R}^{-n} \sup_{\mu\in B_1,\ |\mu|(W_P)=1} \mu P^n W_P \leq \frac{\theta_{0,R}}{\theta_{0,R}-\gamma}$$

and $\Theta_n \le \frac{\gamma^n/\theta_{0,R}^{n-1}}{\theta_{0,R}-\gamma}$ and $\theta_n \le \gamma^n/\theta_{0,R}^n$ (using the notations of Proposition A.3).

By Lemma 2.6, the operator $\mathfrak{R} : B_2 \to B_2$ satisfies Assumption (H) form the Appendix with $\mathfrak{J}_{\mathfrak{R}} = \mathfrak{J}(x)$,

$$E_{\mathfrak{R}}\mu = \sum_{i\in I_R} \mu(\mathbb{1}_{j_R(\cdot)=\mathfrak{J}_{\mathfrak{R}}}\eta_{R,i})\,\nu_{R,i}$$

and

$$\alpha_{\mathfrak{R},n} = C\left(\alpha_{R,n} + \frac{\mathbb{1}_{\mathfrak{J}_{\mathfrak{R}}\ge 1}}{n}\right).$$

We thus deduce from Proposition A.3 that $\mathfrak{S}$ restricted to $B$ satisfies Assumption (H) with $\mathfrak{J}_{\mathfrak{S}} = \mathfrak{J}(x)$, for all $\mu \in B$,

$$\begin{aligned}
E_{\mathfrak{S}}\mu &= E_{\mathfrak{R}}\mu + \sum_{k=0}^{\infty} E_{\mathfrak{R}}\mathfrak{Q}\mathfrak{P}^k\mu \\
&= \sum_{i\in I_R}\left(\mu(\mathbb{1}_{j_R(\cdot)=\mathfrak{J}_{\mathfrak{R}}}\eta_{R,i}) + \sum_{k=0}^{\infty}\theta_{0,R}^{-k-1}\mu P^k Q(\mathbb{1}_{j_R(\cdot)=\mathfrak{J}_{\mathfrak{R}}}\eta_{R,i})\right)\nu_{R,i} \\
&= \sum_{i\in I_R}\mathbb{E}_\mu\left(\theta_{0,R}^{-T_{D_2}}\eta_{R,i}(X_{T_{D_2}})\mathbb{1}_{j_R(X_{T_{D_2}})=\mathfrak{J}(x)}\right)\nu_{R,i} \qquad (3.9)
\end{aligned}$$

and there exists a constant $C$ independent of $x \in D_1$ such that

$$\begin{aligned}
\alpha_{\mathfrak{S},n} &= \alpha_{\mathfrak{R},n} + C\sum_{k=0}^{n-1}\alpha_{\mathfrak{R},n-k-1}\theta_k + C\Theta_n + C\sum_{k=0}^{n-1}\frac{\mathfrak{J}_{\mathfrak{S}}\,k}{n}\theta_k \\
&\le C\sum_{k=0}^{n}\left(\alpha_{R,n-k} + \frac{\mathbb{1}_{\mathfrak{J}(x)\ge 1}}{n-k+1} + \mathfrak{J}(x)\frac{k}{n}\right)\left(\frac{\gamma}{\theta_{0,R}}\right)^k \qquad (3.10) \\
&\le \alpha_{S,n} := C\sum_{k=0}^{n}\left(\alpha_{R,n-k} + j_{0,R}\frac{k}{n}\right)\left(\frac{\gamma}{\theta_{0,R}}\right)^k,
\end{aligned}$$

where $\alpha_{R,0} := 1$. Since $\mathfrak{S}^n = \theta_{0,R}^{-n}S_n$, taking $\mu = \delta_x$ in (3.9), we finally deduce that, for all $x \in D$ and all $f \in L^\infty(W_S)$,

$$\left|\theta_{0,R}^{-n}n^{-\mathfrak{J}(x)}S_n f(x) - \sum_{i\in I_R}\mathbb{E}_x\left(\theta_{0,R}^{-T_{D_2}}\eta_{R,i}(X_{T_{D_2}})\mathbb{1}_{j_R(X_{T_{D_2}})=\mathfrak{J}(x)}\right)\nu_{R,i}(f)\right| \le \alpha_{S,n}W_S(x)\|f\|_{W_S}, \qquad (3.11)$$

where we extended $\mathfrak{J}$ to $D_2$ by setting $\mathfrak{J}(x) := j_R(x)$ if $x \in D_2$.

In order to conclude, it remains to prove that $\theta_{0,S} = \theta_{0,R}$ and that $j_S(x) = \mathfrak{J}(x)$ for all $x \in D$. Inequality (3.11) with $f = \mathbb{1}_D$ implies that $\theta_S(x) \le \theta_{0,R}$ for all $x \in D$, so that $\theta_{0,S} \le \theta_{0,R}$. Moreover, for all $x \in D_2$, $\theta_S(x) = \theta_R(x)$, and thus $\theta_{0,S} \ge \theta_{0,R}$. We deduce that $\theta_{0,S} = \theta_{0,R}$ and hence, using again (3.11) with $f = \mathbb{1}_D$, we deduce that $j_S(x) \le \mathfrak{J}(x)$ for all $x \in D$. On the one hand, for all $x \in D_2$, we have $j_S(x) = j_R(x) = \mathfrak{J}(x)$. On the other hand, for $x \in D_1$, we observe that, for any $n \ge 0$ such that $\mathfrak{J}(x) = j_S(\delta_x P^n Q)$, we have the inequality $\delta_x S^{n+1}\mathbb{1}_D \ge \delta_x P^n Q \mathbb{1}_D$, and hence

$$j_S(x) = j_S(\delta_x S^{n+1}) \ge j_S(\delta_x P^n Q) = \mathfrak{J}(x).$$

We thus proved that $j_S(x) \ge \mathfrak{J}(x)$ for all $x \in D$, which concludes the proof of (ii).

**Under Assumption (A3).** If $x \in D_2$, then (2.3) holds true with $I_S = I_R$ and $\eta_{S,i}(x) = \eta_{R,i}(x)$, $j_S(x) = j_R(x)$ and $\alpha_{S,n} = \alpha_{R,n}$.

Fix $x \in D_1$. We consider $\mathfrak{P}$, $\mathfrak{R}$ and $\mathfrak{S}$ restricted to the Banach space

$$B = B_1 \oplus B_2 \subset \mathcal{M}(W_S),$$

where

$$B_1 = \mathcal{M}(W_P) \text{ and } B_2 = \left\{\mu \in \mathcal{M}(W_R),\ j_R(|\mu|) \le \ell_*\right\}$$

Note that it follows Proposition 2.1 and the assumption that $\mathbb{P}_X(j_R(X_{T_{D_2}}) \le \ell_*) = 1$ that $\mathfrak{Q}B_1 \subset B_2$, and from the rest of Assumption (A3) that $\mathfrak{P} : B_1 \to B_1$, $\mathfrak{R} : B_2 \to B_2$ and $\mathfrak{Q} : B_1 \to B_2$ are bounded operators.

As in the previous step, the operator $\mathfrak{R}$ satisfies Assumption (H) in the Appendix with $\mathfrak{J}_{\mathfrak{R}} = \ell_*$,

$$E_{\mathfrak{R}}\mu = \sum_{i \in I_R} \mu(\mathbb{1}_{j_R(\cdot)=\ell_*}\eta_{R,i})\,\nu_{R,i},$$

and

$$\alpha_{\mathfrak{R},n} = C\left(\alpha_{R,n} + \frac{\mathbb{1}_{\ell_* \ge 1}}{n}\right),$$

for some constant $C > 0$. Moreover, Assumption (A) for $P_n$ implies that $\mathfrak{P}$ satisfies Assumption (H) with $\mathfrak{J}_{\mathfrak{P}} = 0$, $E_{\mathfrak{P}}\mu = \sum_{j \in I_P} \mu(\eta_{P,j})\nu_{P,j}$ and $\alpha_{\mathfrak{P},n} = \alpha_{P,n}$. We conclude from Proposition A.4 that $\mathfrak{S}$ satisfies Assumption (H1) with $\mathfrak{J}_{\mathfrak{S}} = 1 + \ell_*$,

$$E_{\mathfrak{S}}\mu = \frac{1}{\mathfrak{J}_{\mathfrak{S}}} E_{\mathfrak{R}}\mathfrak{Q}E_{\mathfrak{P}}\mu = \frac{\theta_{0,P}^{-1}}{1+\ell_*}\sum_{i \in I_R}\sum_{k \in I_P} \mu(\eta_{P,k})\mathbb{E}_{\nu_{P,k}}\left(\mathbb{1}_{j_R(X_1)=\ell_*}\eta_{R,i}(X_1)\right)\nu_{R,i},$$

and

$$\alpha_{\mathfrak{S},n} = \frac{C}{n}\left(\mathfrak{I}_{\mathfrak{R}} + \sum_{k=0}^{n}\alpha_{\mathfrak{P},k} + \left(\max_{k\geq 0}\alpha_{\mathfrak{P},k}+1\right)\sum_{k=0}^{n}\alpha_{\mathfrak{R},n-k}\left(\frac{n-k}{n}\right)^{\mathfrak{I}_{\mathfrak{R}}}\right)$$
$$\leq \alpha_{S,n} := \frac{C}{n}\left(\ell_* + \sum_{k=0}^{n}\left(\alpha_{P,k}+\alpha_{R,k}\frac{k^{\ell_*}}{n^{\ell_*}}\right)\right),$$

with $\alpha_{P,0}=\alpha_{R,0}=1$. Since $\mathfrak{S}^n\mu = \theta_{0,P}^{-n}\mu S_n$, we deduce that, for all $x\in D_1$ and all $f\in L^\infty(W_S)$,

$$\left|\theta_{0,P}^{-n} n^{-(1+\ell^*)} S_n f(x) - \frac{\theta_{0,P}^{-1}}{1+\ell_*}\sum_{i\in I_R}\sum_{k\in I_P}\eta_{P,k}(x)\mathbb{E}_{\nu_{P,k}}\left(\mathbb{1}_{j_R(X_1)=\ell_*}\eta_{R,i}(X_1)\right)\nu_{R,i}(f)\right|$$
$$\leq \alpha_{S,n}W_S(x)\|f\|_{W_S}. \quad (3.12)$$

This implies that $\theta_S(x)\leq\theta_{0,P}$ for all $x\in D_1$, so $\theta_{0,S}\leq\theta_{0,P}\vee\theta_{0,R}=\theta_{0,P}=\theta_{0,R}$. Conversely, since $Sf\geq Rf$ for all positive $f$, we have $\theta_{0,S}\geq\theta_{0,R}$. We thus deduce that $\theta_{0,S}=\theta_{0,P}=\theta_{0,R}$. We also have, by definition of $S$, $j_S(x)=j_R(x)$ for all $x\in D_2$. Moreover, for all $x\in D_1$, (3.12) implies that $j_S(x)\leq 1+\ell^*$. Note that, up to now, we have neither used the assumption that $\eta_P>0$, nor the second part of (3.3). Therefore, we have proved Corollary 3.2.

It remains to prove that $j_S(x)\geq 1+\ell_*$ for all $x\in D_1$. Fix $x\in D_1$ until the end of the proof. Since $\eta_P(x)>0$, we deduce from Proposition 2.5 (i) that

$$\nu := \frac{1}{\eta_P(x)}\sum_{k\in I_P}\eta_{P,k}(x)\nu_{P,k}$$

is a quasi-stationary distribution for the semigroup $(P_n)_{n\geq 0}$ with exponential convergence parameter $\theta_{0,P}$. Let us first prove that $j_R(\nu Q)=\ell_*$. Since $(R_n)_{n\geq 0}$ satisfies (A), we have for all $y\in D_2$ such that $j_R(y)\leq\ell_*$

$$\theta_{0,R}^{-n}n^{-\ell^*}\delta_y R_n\mathbb{1}_{D_2}\xrightarrow[n\to+\infty]{}\begin{cases}\sum_{i\in I_R}\eta_{R,i}(y) & \text{if } j_R(y)=\ell^*,\\ 0 & \text{if } j_R(y)<\ell_*,\end{cases}$$

where the convergence holds in $L^\infty(W_R)$. Therefore,

$$\theta_{0,R}^{-n}n^{-\ell^*}\nu Q R_n\mathbb{1}_{D_2}\xrightarrow[n\to+\infty]{}\sum_{i\in I_R}\nu Q\left(\eta_{R,i}\mathbb{1}_{j_R(\cdot)=\ell_*}\right). \quad (3.13)$$

Now, using that $\nu$ is a quasi-stationary distribution,

$$\begin{aligned}\mathbb{E}_\nu\left(\eta_R(X_{T_{D_2}})\mathbb{1}_{j_R(X_{T_{D_2}})=\ell_*}\right) &= \sum_{m=0}^{+\infty}\nu P_m Q\left(\eta_R(\cdot)\mathbb{1}_{j_R(\cdot)=\ell_*}\right)\\ &= \frac{1}{1-\theta_{0,P}}\nu Q\left(\eta_R(\cdot)\mathbb{1}_{j_R(\cdot)=\ell_*}\right).\end{aligned}$$

By the second part of Assumption (3.3), the left-hand side is positive, so we have proved that $\theta_{0,R}^{-n}n^{-\ell^*}\nu QR_n\mathbb{1}_{D_2}$ converges to a positive limit. This shows that $j_R(\nu Q)=\ell_*$.

For all $n\geq 1$, using the fact that $S_n=P_n+R_n+\sum_{k=0}^{n-1}P_{n-k-1}QR_k$, we have

$$\begin{aligned}n^{-(\ell_*+1)}\theta_0^{-n}\nu S_n\mathbb{1}_D &\geq n^{-(\ell_*+1)}\theta_0^{-n}\sum_{k=0}^{n-1}\nu P_{n-k-1}QR_k\mathbb{1}_{D_2}\\ &= n^{-(\ell_*+1)}\theta_0^{-1}\sum_{k=1}^{n-1}\theta_0^{-k}\nu QR_k\mathbb{1}_{D_2}\\ &= n^{-(\ell_*+1)}\theta_0^{-1}\sum_{k=1}^{n-1}k^{\ell_*}\left[k^{-\ell_*}\theta_0^{-k}\nu QR_k\mathbb{1}_{D_2}\right].\end{aligned}$$

Using that, by (3.13), $k^{-\ell_*}\theta_0^{-k}\nu QR_k\mathbb{1}_{D_2}$ converges to a positive limit when $k\to+\infty$, we deduce that

$$\liminf_{n\to+\infty}n^{-(\ell_*+1)}\theta_0^{-n}\nu S_n\mathbb{1}_D>0.$$

This shows that $j_S(\nu)\geq\ell_*+1$. Finally, we have $\theta_0^{-n}\delta_x P_n\to\sum_{k\in I_P}\eta_{P,k}(x)\nu_{P,k}=\eta_P(x)\nu$ in $\mathcal{M}(W_S)$ when $n\to+\infty$. Since $\eta_P(x)>0$, we deduce from the lower semi-continuity of $j_S$ (see Proposition 2.1) that

$$\liminf_{n\to+\infty}j_S(\theta_0^{-n}\delta_x P_n)\geq j_S(\eta_P(x)\nu)=j_S(\nu)=\ell_*+1.$$

Using again Proposition 2.1, we have $j_S(x)=j_S(\theta_0^{-n}\delta_x S_n)\geq j_S(\theta_0^{-n}\delta_x P_n)$ for all $n\geq 0$. So we finally deduce that

$$j_S(x)=\liminf_{n\to+\infty}j_S(\theta_0^{-n}\delta_x S_n)\geq\liminf_{n\to+\infty}j_S(\theta_0^{-n}\delta_x P_n)\geq j_S(\nu_P)=\ell_*+1.$$

This concludes the proof of Theorem 3.1. □

# 4 Quasi-stationary distributions in reducible state spaces

Our goal is to study quasi-stationary distributions on general reducible state spaces, a situation which naturally leads to non-zero polynomial convergence parameters. In particular, we extend the results of [10, Section 6.2], which are stated under conditions ensuring that the polynomial convergence parameter of the process vanishes. We refer the reader to [27, 28] where an in-depth study of the quasi-stationary distributions on finite reducible state spaces has been conducted (see also the survey [29], an earlier work [22] resumed in [13, Section 9], and the more recent works [7, 5]). The quasi-stationary distribution of particular processes on reducible state spaces with finitely many communication classes have also previously been studied in [26] (for multi-type Galton-Watson processes), [17, Section 3] (for discrete state space processes, under conditions ensuring that the polynomial convergence parameter vanishes), [8] (for multitype Dawson-Watanabe processes).

We consider a Markov process $X$ with semigroup $S$ on a general state space $D$ that can be decomposed into finitely many disjoint sets $E_1, E_2, \dots, E_k$, where $k \geq 2$. We denote, for all $i \in \{1, \dots, k\}$, by $Y^{(i)}$ the process

$$Y_n^{(i)} = \begin{cases} X_n & \text{if } n < T_{\cup_{j\neq i} E_j \cup \{\partial\}}, \\ \partial & \text{otherwise} \end{cases}$$

and define by $\theta_{0,i}$ its exponential convergence parameter. We let

$$\bar{\theta} := \max_{j \in \{1,\dots,k\}} \theta_{0,j}$$

and assume that, for all $i \in \{1, \dots, k\}$ such that $\theta_{0,i} < \bar{\theta}$, there exists $\gamma < \bar{\theta}$ and a function $W_i \geq 1$ such that, for all $x \in E_i$ and for some constant $C > 0$,

$$\mathbb{E}_x(W_i(Y_n^{(i)})) \leq C\gamma^n W_i(x), \ \forall n \geq 0. \tag{4.1}$$

Note that (4.1) implies that $\theta_{0,i} \leq \gamma$ and that the constant $\gamma < \bar{\theta}$ can (and will) be chosen independent of $i$ without loss of generality. Assume also that, for all $i \in \{1, \dots, k\}$ such that $\theta_{0,i} = \bar{\theta}$, the process $Y^{(i)}$ satisfies Assumption (A) with the objects $\theta_{0,i}$, $j_i$, $\alpha_{i,n}$, $W_i$, $\nu_i$ and $\eta_i$ such that $j_i \equiv 0$, $I_i = \{1\}$ and $\eta_i > 0$ on $E_i$ (note that we omit the second index for $\eta_{i,1}$ and $\nu_{i,1}$). Many references provide practical criteria to check Assumption (A) with $I_S = \{1\}$, $j_S \equiv 0$, $\eta_S > 0$ and with $\alpha_{S,n}$ converging exponentially fast to 0, which corresponds to the classical irreducible situation, see for example [16, 9, 10, 11, 15, 20, 1, 18, 19, 21, 4].

The process $Y^{(i)}$ is called the process $X$ restricted to $E_i$. More generally, for all $M \subset D$, we call process $X$ restricted to $M$ the process $X$ killed after its first exit time from $M$. We set $W = \sum_{i=1}^k W_i$ and assume that there exists a constant $C > 0$ such that

$$\mathbb{E}_x(W(X_1)) \leq CW(x), \quad \forall x \in D. \tag{4.2}$$

We assume that the set $\{1,\dots,k\}$ can be equiped with a partial order $\prec$ such that $i \prec j$ if and only if $E_i$ is accessible from $E_j$ in the sense that: for all $i \neq j$, if $i \prec j$, then

$$\forall x \in E_j, \quad \mathbb{P}_x(T_{E_i} < +\infty) > 0 \tag{4.3}$$

and, if $i \not\prec j$, then

$$\forall x \in E_j, \quad \mathbb{P}_x(T_{E_i} < +\infty) = 0.$$

We also assume that the oriented graph $G$ induced by the partial order $\prec$ on $\{1,\dots,k\}$ is connected, since otherwise the dynamics of $X$ in distinct connected components of this graph are unrelated and can be studied using the result below separately. The result below can also be applied to $X$ restricted to connected subgraphs of $G$, to provide convergence results for specific initial conditions and test functions (see for example Remark 16 below).

To state this result, we introduce the following notations. We first define the sets

$$\overline{F} := \left\{ i \in \{1,\dots,k\} : \theta_{0,i} = \bar{\theta} \right\}$$

and

$$\overline{F}_0 := \left\{ \text{minimal elements in } \overline{F} \text{ for the order } \prec \right\}.$$

For all $i_0 \in \{1,\dots,k\}$, we define the set of paths from $i_0$ to $\overline{F}$ as

$$\mathscr{P}_{i_0} = \left\{ (i_0,\dots,i_p),\ p \geq 0,\ i_p \in \overline{F},\ i_{\ell-1} \succ i_\ell\ \forall \ell \in \{1,\dots,p\} \right\}$$

and the set $\overline{\mathscr{P}}_{i_0}$ of complete paths from $i_0$ to $\overline{F}$ as the set of maximal elements of $\mathscr{P}_{i_0}$ for the partial order induced by the inclusion of paths. We then define

$$j_{i_0} := \max_{(i_0,\dots,i_p)\in\overline{\mathscr{P}}_{i_0}} \#\left( \{i_0,\dots,i_{p-1}\} \cap \overline{F} \right), \tag{4.4}$$

with the convention that $\max \emptyset = 0$.

**Theorem 4.1.** *Under the above assumptions, the process X satisfies Condition (A) with*

$$\theta_{0,S} = \bar{\theta}, \quad I_s = \overline{F}_0, \quad W_S = \sum_{i=1}^{k} W_i,$$
$$j_S(x) = j_{i_0} \textit{ for all } x \in E_{i_0} \textit{ and } i_0 \in \{1,\dots,k\},$$

*and, for all* $i \in I_s$,

$$\nu_{S,i} \propto \nu_i + \sum_{\ell \geq 0} \theta_{0,S}^{-\ell-1} \mathbb{P}_{\nu_i}(X_1 \not\in E_i,\ X_{\ell+1} \in \cdot) \tag{4.5}$$

*and*

$$\eta_{S,i}(x) > 0 \textit{ for all } x \in E_{i_0} \textit{ with } i_0 \in \{1,\dots,k\} \textit{ s.t. } i_0 \succ i \tag{4.6}$$

*and*

$$\eta_{S,i}(x) = 0 \textit{ for all } x \in E_{i_0} \textit{ with } i_0 \in \{1,\dots,k\} \textit{ s.t. } i_0 \not\succ i. \tag{4.7}$$

*Remark* 14. In this theorem, the functions $\eta_{S,i}$ can also be expressed in terms of the parameters of the problem, since they are constructed in the proof below with an inductive argument, with explicit expressions at each step. △

*Remark* 15. Similarly, the speed of convergence $\alpha_{S,n}$ is also constructed explicitely with an inductive argument in the proof below. In particular, if it is assumed that $\alpha_{i,n}$ converges exponentially fast to 0 for all $i$ such that $\theta_{0,i} = \bar{\theta}$, one can easily check that $\alpha_{S,n}$ also converges to 0 exponentially fast if $j_S \equiv 0$, and converges to 0 polynomially in $O(1/n)$ otherwise. △

*Remark* 16. It follows from the last theorem and Corollary 2.4 that the set of quasi-stationary distributions $\nu$ for $X$ such that $\nu(W_S) < +\infty$ and $\nu(\eta_S) > 0$ has dimension $\#\overline{F}_0$ and is spanned (in the sense of convex hulls) by $\nu_{S,i}$, $i \in \overline{F}_0$. Our result also allows to characterize all quasi-stationary distributions $\nu$ of $X$ such that $\nu(W_S) < \infty$: one can obtain the other quasi-stationary distributions by applying Theorem 4.1 to the process $X$ restricted to

$$\bigcup_{i \in \{1,\dots,k\} : \overline{\mathscr{P}}_i = \emptyset} E_i$$

and by proceeding recursively. All the new quasi-stationnary distributions obtained this way have an exponential convergence parameter (strictly) smaller than $\bar{\theta}$.

In the particular case where the state space $D$ is finite, our result are thus reminiscent of [28] (see in particular Theorems 4.3 and 5.1 therein). These results are already quite complete, and one of our main contributions to the problem in the finite state space situation is to determine explicitly the polynomial convergence parameter associated to each communication classes, and to emphasize the support of the functions $\eta_{S,i}$. △

*Remark* 17. In cases where some of the $\eta_i$, $i \in \bar{F}$, is not positive on its domain $E_i$, one could obtain a similar result using Corollary 3.2 instead of Theorem 3.1 (iii) in the proof below. For this, Assumption (4.3) needs to be replaced by

$$\forall i, j \in \{1, \dots, k\} \text{ such that } i \prec j,\ \forall x \in E_j, \quad \sup_{n \geq 0} \mathbb{E}_x(\eta_i(X_n)) > 0$$

to make the induction argument used in the proof below work. Then, one obtains the same expressions for $\eta_{S,i}$ and $\nu_{S,i}$ for the process $X$ in (2.3), but with a non-optimal exponent $j$ instead of $j_S$. However, as noticed in Remark 13, we have $j(x) = j_S(x)$ for all $x$ such that $\eta_S(x) = \sum_{i \in I_S} \eta_{S,i} > 0$. It is easy to check from the proof below that $\eta_S(x) > 0$ for all $x \in D$ such that there exists $i \in \overline{F}_0$ satisfying $\mathbb{P}_x(T_{E_i} < \infty, \eta_i(X_{T_{E_i}}) > 0) > 0$. △

*Remark* 18. The above result also allows to study reducible processes with denumerably many communication classes, provided that the quantity $j_{i_0}$ defined in (4.4) is finite for all $i_0$. In particular, each $E_i$, $i \notin \overline{F}$, may contain several communication classes. △

*Proof of Theorem 4.1.* For all $\ell \geq 0$, we define by induction

$$\overline{F}_{\ell+1} := \left\{ \text{minimal elements in } \overline{F} \setminus \left( \overline{F}_0 \cup \ldots \cup \overline{F}_\ell \right) \text{ for the order } \prec \right\}.$$

Note that $\overline{F}_\ell = \emptyset$ iff $\ell > \bar{\ell}$ for some $\bar{\ell} < +\infty$. We define by decreasing induction on $\ell \leq \bar{\ell}$,

$$\overline{J}_{\bar{\ell}} = \left\{ j \in \{1, \dots, k\} : \exists i \in \overline{F}_{\bar{\ell}}, j \succeq i \right\}$$

and

$$\overline{J}_\ell = \left\{ j \in \{1, \dots, k\} \setminus \left( \overline{J}_{\ell+1} \cup \ldots \cup \overline{J}_{\bar{\ell}} \right) : \exists i \in \overline{F}_\ell, j \succeq i \right\},$$

where $j \succeq i$ means that $j \succ i$ or $j = i$. For all $\ell \leq \bar{\ell}$, we also define

$$\overline{E}_\ell := \bigcup_{i \in \overline{J}_\ell} E_i.$$

Observe that, for all $\ell \in \{0,\dots,\bar{\ell}\}$ and all $i \in \overline{J}_\ell$, we have $\ell = j_i$, as defined in (4.4).

**Step 1.** For all $\ell \le \bar{\ell}$, let $S^{(\ell)}$ be the semigroup of the process $X$ restricted to $\overline{E}_\ell$. Let us prove that $S^{(\ell)}$ satisfies Condition (A) with $\theta_{0,S^{(\ell)}} = \bar{\theta}$, $I_{S^{(\ell)}} = \overline{F}_\ell$, $W_{S^{(\ell)}} = W$, $j_{S^{(\ell)}} \equiv 0$ and, for all $j \in \overline{F}_\ell$, $\eta_{S^{(\ell)},j} > 0$ on $E_i$ for all $i \in \overline{J}_\ell$ such that $i \succeq j$. In addition, $\nu_{S^{(\ell)},j} = \nu_j$ for all $j \in \overline{F}_\ell$.

Fix $i \in \overline{F}_\ell \subset \overline{J}_\ell$. Then $i \not\succeq j$ for all $j \in \overline{J}_\ell \setminus \{i\}$ and hence the process $X$ with initial position in $E_i$ and restricted to $\overline{E}_\ell$ equals the process $X$ restricted to $E_i$. In particular, by assumption, for all $x \in E_i$,

$$\left|\bar{\theta}^{-n}\mathbb{E}_x(f(X_n)\mathbb{1}_{n<T_{\{\partial\}\cup D\setminus\overline{E}_\ell}}) - \eta_i(x)\nu_i(f)\right| \le \alpha_{i,n} W(x)\|f\|_W,$$

with $\eta_i$ positive on $E_i$. We deduce that, for all $x \in \bigcup_{i\in\overline{F}_\ell} E_i$,

$$\left|\bar{\theta}^{-n}\mathbb{E}_x(f(X_n)\mathbb{1}_{n<T_{\{\partial\}\cup D\setminus\overline{E}_\ell}}) - \sum_{i\in\overline{F}_\ell} \eta_i(x)\nu_i(f)\right| \le \sum_{i\in\overline{F}_\ell} \alpha_{i,n} W(x)\|f\|_W.$$

Now let $i_1$ be a minimal element in $\overline{J}_\ell \setminus \overline{F}_\ell$. Then the process $X$ with initial position in $E_{i_1}$ and restricted to $\overline{E}_\ell$ equals $Y^{(i)}$, the process $X$ restricted to $\bigcup_{j\in\{i_1\}\cup\overline{F}_\ell} E_j$. Hence, by (4.1) and according to Theorem 3.1 (ii), we have, for all $x \in E_{i_1}$,

$$\left|\bar{\theta}^{-n}\mathbb{E}_x(f(X_n)\mathbb{1}_{n<T_{\{\partial\}\cup D\setminus\overline{E}_\ell}}) - \sum_{j\in\overline{F}_\ell} \eta'_{i_1,j}(x)\nu_j(f)\right| \le \alpha'_{i_1,n} W(x)\|f\|_W,$$

where $\alpha'_{i_1,n} \to 0$ when $n \to +\infty$, and where, for all $j \in \overline{F}_\ell$,

$$\eta'_{i_1,j}(x) \propto \mathbb{E}_x\left(\bar{\theta}^{-T_{\overline{E}_\ell}}\eta_j(X_{T_{\overline{F}_\ell}})\right).$$

We observe in particular that, for each $j \in \overline{F}_\ell$ such that $i_1 \succ j$, $\eta'_{i_1,j}(x)$ is positive since $\eta_j$ is positive on $E_j$, since $T_{\overline{F}_\ell} = T_{E_j}$ $\mathbb{P}_x$-almost surely on the event $T_{E_j} < +\infty$, and since, by definition, $\mathbb{P}_x(T_{E_j} < +\infty) > 0$.

Proceeding similarly and by induction on the minimal elements of $\overline{J}_\ell \setminus (\overline{F}_\ell \cup \{i_1\})$, we deduce that there exist non-negative, non-zero functions $\overline{\eta}_{\ell,j}$ on $E_j$, $j \in \overline{F}_\ell$, such that, for all $x \in \overline{E}_\ell$,

$$\left|\bar{\theta}^{-n}\mathbb{E}_x(f(X_n)\mathbb{1}_{n<T_{\{\partial\}\cup D\setminus\overline{E}_\ell}}) - \sum_{j\in\overline{F}_\ell} \overline{\eta}_{\ell,j}(x)\nu_j(f)\right| \le \overline{\alpha}_{\ell,n} W(x)\|f\|_W,$$

where $\overline{\alpha}_{\ell,n} \to 0$ when $n \to +\infty$, and such that, for all $i \in \overline{J}_\ell$ and $j \in \overline{F}_\ell$ such that $i \succ j$, $\overline{\eta}_{\ell,j}$ is positive on $E_i$. This entails the claim made at the beginning of Step 1.

**Step 2.** Let $\bar{S}$ be the semigroup of the process $X$ restricted to $\bigcup_{\ell=0}^{\bar{\ell}} \overline{E}_\ell$. Let us prove that $\bar{S}$ satisfies Condition (A) with $\theta_{0,\bar{S}} = \bar{\theta}$, $W_{\bar{S}} = W$ and $j_{\bar{S}} = \ell$ on $\overline{E}_\ell$ for all $\ell \le \bar{\ell}$. In addition, we prove that $I_{\bar{S}} = \overline{F}_0$ and, for all $j \in \overline{F}_0$, $\eta_{\bar{S},j} > 0$ on $E_i$ for all $i \in \bigcup_{\ell=1}^{\bar{\ell}} \overline{J}_\ell$ such that $i \succeq j$, and $\nu_{\bar{S},j} = \nu_j$.

Let $\bar{S}^{(\ell)}$ be the semigroup of the process restricted to $\bigcup_{\ell'=0}^{\ell} \overline{E}_{\ell'}$. We prove by induction on $\ell$ that, for all $\ell \in \{0,\dots,\bar{\ell}\}$, $\bar{S}^{(\ell)}$ satisfies Assumption (A) with $\theta_{0,\bar{S}^{(\ell)}} = \bar{\theta}$, $W_{\bar{S}^{(\ell)}} = W$ and $j_{\bar{S}^{(\ell')}} = \ell'$ on $\overline{E}_{\ell'}$ for all $\ell' \le \ell$, and that $I_{\bar{S}^{(\ell)}} = \overline{F}_0$ and, for all $j \in \overline{F}_0$, $\eta_{\bar{S}^{(\ell)},j} > 0$ on $E_i$ for all $i \in \bigcup_{\ell'=0}^{\ell} \overline{J}_{\ell'}$ such that $i \succeq j$, and $\nu_{\bar{S}^{(\ell)},j} = \nu_j$.

Since $\bar{S}^{(0)} = S^{(0)}$, we deduce from Step 1 that the property holds true with $\ell = 0$. Assume now that the property holds true for some $\ell \in \{0,\dots,\bar{\ell}-1\}$ and let us show that it holds true for $\ell+1$. We denote by $P$ the semigroup of the process restricted to $D_1 := \overline{E}_{\ell+1}$, by $Q$ the transition kernel of the process from $D_1 = \overline{E}_{\ell+1}$ to $D_2 := \bigcup_{\ell'=0}^{\ell} \overline{E}_{\ell'}$, and by $R$ the semigroup of the process restricted to $D_2 = \bigcup_{\ell'=0}^{\ell} \overline{E}_{\ell'}$. In particular $S = P + Q + R$ and our aim is to show that Assumption (A3) holds true, then to apply Theorem 3.1. By construction, $i \not\succeq j$ for all $i \in \bar{J}_{\ell'}$ with $\ell' \le \ell$ and $j \in \bar{J}_{\ell+1}$, so that $\mathbb{P}_x(T_{D_1} = +\infty) = 1$ for all $x \in D_2$. According to Step 1, the semigroup $P$ satisfies Assumption (A) with $j_{0,P} = 0$, $\theta_{0,P} = \bar{\theta}$ and $\eta_P > 0$. According to the induction assumption, the semigroup $R$ satisfies Assumption (A3) with $j_{0,R} < +\infty$ and $\theta_{0,R} = \bar{\theta}$. In addition, setting $W_R := \sum_{\ell' \le \ell} \sum_{i \in \bar{J}_{\ell'}} W_i$ and $W_P = C \sum_{i \in \bar{J}_{\ell+1}} W_i$, we deduce from (4.2) that, for all $x \in D_1$,

$$\mathbb{E}_x(W_R(X_1)) \le W_P(x).$$

We set $\ell_* = \ell$. Then, according to the induction assumption, $j_R \le \ell_*$ on $D_2$, so that, for all $x \in D_1$,

$$\mathbb{P}_x(T_{D_2} < +\infty \text{ and } j_R(X_{T_{D_2}}) \le \ell_*) = \mathbb{P}_x(T_{D_2} < +\infty).$$

Finally, by definition of the sets $\overline{F}_{\ell+1}$ and $\overline{J}_{\ell+1}$, for all $i \in \overline{J}_{\ell+1}$, there exists $j \in \overline{J}_\ell$ such that $i \succ j$. In particular, for all $x \in D_1$, there exists $j \in \overline{J}_\ell$ such that $\mathbb{P}_x(T_{E_j} < +\infty) > 0$ and, since by the induction assumption, $j_R \equiv \ell = \ell_*$ on $\overline{E}_\ell$ and $\eta_R > 0$ on $\overline{E}_\ell$, we deduce that, for all $x \in D_1$, $\mathbb{P}_x(T_{D_2} < +\infty, j_R(X_{T_{D_2}}) = \ell_*, \eta_R(X_{T_{D_2}}) > 0) > 0$. Hence we can apply Theorem 3.1 (iii) and deduce that Assumption (A) holds true for $\bar{S}^{(\ell+1)}$ with $\theta_{0,\bar{S}^{(\ell+1)}} = \theta_{0,R} = \bar{\theta}$, with $W_{\bar{S}^{(\ell+1)}} = W_R + W_P$ and hence also with $W_{\bar{S}^{(\ell+1)}} = W$, and $j_{\bar{S}^{(\ell+1)}} = \ell_* + 1 = \ell + 1$ on $\overline{E}_{\ell+1}$ and $j_{\bar{S}^{(\ell+1)}} = j_R = \ell'$ on $\overline{E}_{\ell'}$ for all $\ell' \le \ell$, and that $I_{\bar{S}^{(\ell+1)}} = I_R = \overline{F}_0$, with $\nu_{\bar{S}^{(\ell)},j} = \nu_{R,j} = \nu_j$ for all $j \in \overline{F}_0$. In addition, fixing $j \in \overline{F}_0$, the induction assumption and the expression of $\eta_{S,j}$ in Theorem 3.1 (iii) entails that, for all $i \in \bigcup_{\ell'=0}^{\ell}$ such that $i \succ j$, $\eta_{\bar{S}^{(\ell+1)},j} > 0$ on $E_i$. This concludes the induction step and shows that the property holds true with $\ell = \bar{\ell}$, which concludes the proof of Step 2.

**Step 3.** Let us now consider the set $G := \{1, \dots, k\} \setminus \cup_{\ell=1}^{\bar{\ell}} \overline{J}_\ell$. We define $G_0$ as the minimal elements in $G$ for the order $\succ$, and, by induction,

$$G_{\ell+1} = \{\text{minimal elments in } G \setminus (G_0 \cup \dots \cup G_\ell) \text{ for the order } \succ\}$$

We define $\bar{\ell}'$ as the maximal index $\ell$ such that $G_\ell \neq \emptyset$. We first show that, for all $x \in \cup_{i \in G} E_i$ and all $n \ge 0$,

$$\mathbb{E}_x(W(X_n)) \le C' \bar{\gamma}^n W(x), \tag{4.8}$$

for some constants $C' > 0$ and $\bar{\gamma} \in (\gamma, \bar{\theta})$. We proceed by induction on the sets $G_\ell$. The property holds true on $G_0$ by (4.1). Assume that it holds true for all $x \in G_{\ell'}$, $\ell' \le \ell$, for some $\ell \le \bar{\ell}' - 1$. Then, using Markov's property, we have for all $i \in G_{\ell+1}$ and $x \in E_i$,

$$\mathbb{E}_x(W(X_n)) = \mathbb{E}_x(\mathbb{1}_{T_{E_i^c} > n} W_i(X_n)) + \sum_{k=1}^{n} \mathbb{E}_x(\mathbb{1}_{T_{E_i^c} = k} \mathbb{E}_{X_k}(W(X_{n-k}))),$$

where $T_{E_i^c}$ is the exit time from $E_i$. By definition of $G_{\ell+1}$, $X_{T_{E_i^c}}$ belongs to $\{\partial\} \cup \bigcup_{\ell' \le \ell} \bigcup_{i \in G_{\ell'}} E_i$, and hence, using the induction assumption,

$$\mathbb{1}_{T_{E_i^c} = k} \mathbb{E}_{X_k}(W(X_{n-k})) \le C' \, \mathbb{1}_{T_{E_i^c} = k} \bar{\gamma}^{n-k} W(X_k).$$

Hence, using in addition (4.1) for the first term in the last equation,

$$\begin{aligned}
\mathbb{E}_x(W(X_n)) &\le C\gamma^n W(x) + \sum_{k=1}^{n} C' \bar{\gamma}^{n-k} \mathbb{E}_x(\mathbb{1}_{T_{E_i^c} = k} W(X_k)) \\
&\le C\gamma^n W(x) + \sum_{k=1}^{n} C' \bar{\gamma}^{n-k} \mathbb{E}_x(\mathbb{1}_{T_{E_i^c} > k-1} \mathbb{E}_{X_{k-1}}(W(X_1))) \\
&\le C\gamma^n W(x) + \sum_{k=1}^{n} C' \bar{\gamma}^{n-k} \mathbb{E}_x(\mathbb{1}_{T_{E_i^c} > k-1} C W_i(X_{k-1})),
\end{aligned}$$

where we used (4.2). Then using again (4.1), we deduce that

$$\mathbb{E}_x(W(X_n)) \le C\gamma^n W(x) + \sum_{k=1}^{n} C'\bar{\gamma}^{n-k} C^2 \gamma^k W(x).$$

Since $\gamma < \bar{\theta}$, we deduce that, up to a change in the constants $C'$ and $\bar{\gamma}$, the property holds true for $\ell+1$. This shows by induction that (4.8) holds true.

One immediately checks that Assumption (A1) holds true with $P$ the semigroup of $X$ restricted to $D_1 := \bigcup_{\ell=0}^{\bar{\ell}} \overline{E}_\ell$ (this is Step 2) and $R$ the semigroup of $X$ restricted to $D_2 := \bigcup_{\ell=0}^{\bar{\ell}'} \bigcup_{i\in G_\ell} E_\ell$. Hence, using Theorem 3.1 (i), we deduce that Assumption (A) holds true for the process $X$ on $D = D_1 \cup D_2$, with $\theta_{0,S} = \theta_P = \bar{\theta}$, $I_S = I_P = \overline{F}_0$, $W_S = W_P + W_R$ and hence for $W_S = W$, for $j_S \equiv 0$ on $D_2$ and $j_S = j_P$ on $D_1$. Using the fact that $j_{i_0} = 0$ for all $i_0 \in G$ and Step 2 for $i_0 \notin G$, we deduce that $j_S(x) = j_{i_0}$ for all $x \in E_{i_0}$ and $i_0 \in \{1, \dots, k\}$. In addition, for all $j \in \overline{F}_0$ and all $x \in E_i$ with $i \succ j$, $\eta_{S,j}(x) \propto \eta_{P,i}(x) > 0$ (according to Step 2), while, for all $x \in D_2$, $\eta_{S,j}(x) = 0$, which corresponds exactly to the last two assertion (4.6) and (4.7) of Theorem 4.1. It remains to prove (4.5). Theorem 3.1 (i) entails that, for all $i \in \overline{F}_0$,

$$\begin{aligned} \nu_{S,i} &\propto \nu_{P,i} + \sum_{k\ge 0} \theta_{0,S}^{-k-1} \mathbb{P}_{\nu_{P,i}}\left(T_{D_2} = 1,\ X_{k+1} \in \cdot\right) \\ &= \nu_i + \sum_{k\ge 0} \bar{\theta}^{-k-1} \mathbb{P}_{\nu_i}\left(T_{D_2} = 1,\ X_{k+1} \in \cdot\right). \end{aligned}$$

But, for all $i \in \overline{F}_0$ and $x \in E_i$, $T_{D_2}$ is almost surely equal to the exit time from $E_1$, and hence (4.5) follows. □

# 5 Lyapunov type criterion for Markov processes in discrete state spaces

Let $X = (X_n, n \in \mathbb{Z}_+)$ be a Markov chain on a discrete state space $D \cup \{\partial\}$, with $\partial$ absorbing. It is well known that, when $X$ is aperiodic and irreducible, that is when $\mathbb{P}_x(\exists n \ge 0,\ X_n = y) > 0$ for all $x, y \in D$, existence of a quasi-stationary distribution is implied by the existence of a Lyapunov type function (see for instance [14, 10], see also [29] for a general account on quasi-stationary distributions for discrete state space models). We show in this section that the irreducibility assumption can actually be removed entirely.

In the following result, we say that $X$ is aperiodic if all states in $D$ are aperiodic (with the usual convention that a state $x \in D$ such that $\mathbb{P}_x(\exists n \geq 0,\ X_n = x) = 0$ is said aperiodic).

**Theorem 5.1.** *Assume that $X$ is aperiodic, that there exists $x_0 \in D$ such that $\mathbb{P}_{x_0}(\exists n \geq 0,\ X_n = x_0) > 0$ and that there exists a function $V : D \to [1, +\infty)$ such that $\{x \in D,\ V(x) \leq C\}$ is finite for all constants $C > 0$ and such that*

$$\frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1<\tau_\partial})}{V(x)} \xrightarrow[V(x)\to+\infty]{} 0. \tag{5.1}$$

*Then Assumption (A) holds true with $W_S = V$ and, in particular, $X$ admits a quasi-stationary distribution. In addition, $\mathbb{P}_{x_0}(X_n \in \cdot \mid n < \tau_\partial)$ converges in $\mathcal{M}(V)$ when $n \to +\infty$ toward a quasi-stationary distribution of $X$.*

*Remark* 19. Despite its generality, the assumption that there exists $x_0 \in D$ such that $\mathbb{P}_{x_0}(\exists n \geq 0,\ X_n = x_0) > 0$ is actually not necessary for the existence of a quasi-stationary distribution. Consider for instance the process with $D = \{1, 2, \ldots\}$ and $\partial = 0$, with almost sure transition from $i$ to $i - 1$ for all $i \geq 1$. Then, choosing $\nu(i) = \frac{\theta}{1-\theta}\theta^i$ for all $i \geq 1$ and any $\theta \in (0, 1)$, we have

$$\mathbb{P}_\nu(X_1 = i) = \nu(i+1) = \frac{\theta}{1-\theta}\theta^{i+1} = \theta\,\nu(i),$$

so that $\nu$ is a quasi-stationary distribution. △

*Remark* 20. In (5.1), we assumed for simplicity that $\frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1<\tau_\partial})}{V(x)} \xrightarrow[V(x)\to+\infty]{} 0$. However, a straightforward adaptation of the proof leads to a finer result: denoting by $C_i$, $i \in I$ with $I = \mathbb{N} := \{1, 2, \ldots\}$ or $I = \{1, \ldots, n\}$ for some $n \geq 1$, the collection of communication classes of the process, and by $\theta_i$ the exponential convergence parameter associated to each $C_i$, it is sufficient to assume that

$$\limsup_{V(x)\to+\infty} \frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1<\tau_\partial})}{V(x)} < \sup_{i\in I}\theta_i.$$

Another natural and straightforward adaptation of the result is to replace by $V$ by any function $V : D \to [1, +\infty)$ without assuming that $\{x \in D,\ V(x) \leq C\}$ is finite for all $C \geq 0$, but such that, for a non-decreasing sequence of finite sets $(K_n)_{n\geq 0}$ such that $D = \cup_n K_n$, we have

$$\limsup_{n\to+\infty} \inf_{x\notin K_n} \frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1<\tau_\partial})}{V(x)} < \sup_{i\in\mathbb{N}}\theta_i.$$

△

*Remark* 21. The aperiodicity assumption is actually not needed for all $x \in D$: one easily checks that it is only required over communication classes whose exponential convergence parameter is maximal. More generally, adaptation of these results to periodic processes is common procedure, and we leave its details to the interested reader. $\triangle$

*Proof of Theorem 5.1.* For all $x \in D$, let $C_x$ be the communication class of $x$, and let $(x_i)_{i \in I}$, where $I$ is either $\mathbb{N}$ or $\{1, \dots, n\}$ for some $n$, be such that $D$ is the disjoint union of the sets $C_{x_i}$, $i \in I$. We take (without loss of generality) $x_1 = x_0$ and write $C_i$ instead of $C_{x_i}$.

Let $i \in I$ be such that $\mathbb{P}_{x_i}(\exists n \geq 0,\ X_n = x_i) > 0$. By assumption, this is the case for $i = 1$. Then the process $X$ restricted to $C_i$ is irreducible and satisfies Assumption (E) in [9] (this is a direct adaptation to the discrete time setting of the proof of Theorem 5.1 in the last reference). In particular (see for instance [11]), the process $X$ restricted to $C_i$ satisfies Assumption (A) with $j_S \equiv 0$, $\eta_S$ positive and $W_S = V_{|C_i}$. We denote by $\theta_i$ the associated exponential convergence parameter.

Let $i \in I$ be such that $\mathbb{P}_{x_i}(\exists n \geq 0,\ X_n = x_i) = 0$. Then $C_i = \{x_i\}$ and

$$\frac{\mathbb{E}_{x_i}(V(X_1)\mathbb{1}_{X_1 \in C_i})}{V(x_i)} = 0.$$

Now, define

$$J := \left\{ i \in I,\ \frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1<\tau_\partial})}{V(x)} < \theta_1\ \forall x \in C_i \right\}.$$

By assumption, there exists only finitely many points $x \in D$ such that $\frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1<\tau_\partial})}{V(x)} \geq \theta_1/2$, and hence there exists $\rho < \theta_1$ such that

$$J = \left\{ i \in I,\ \frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1<\tau_\partial})}{V(x)} \leq \rho\ \forall x \in C_i \right\}$$

In particular, for all $x \in \cup_{j \in J} C_j$,

$$\frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{X_1 \in \cup_{j \in J} C_j})}{V(x)} \leq \rho,$$

so, for all $n \geq 1$,

$$\mathbb{E}_x\left(V(X_n)\mathbb{1}_{T_{\{\partial\}\cup D \setminus \cup_{j \in J} C_j} > n}\right) \leq \rho^n V(x).$$

Let $J' = I \setminus J$. Note that, by assumption, $J'$ is finite. In addition, all $i \in J'$ is either such that $\mathbb{P}_{x_i}(\exists n \geq 0,\ X_n = x_i) > 0$ or such that $\mathbb{P}_{x_i}(\exists n \geq 0,\ X_n = x_i) = 0$, hence the process restricted to $C_i$ either satisfies Assumption (A) with $j_S \equiv 0$, $\eta_S > 0$ and $W_S = V_{|C_i}$, or satisfies

$$\sup_{x \in C_i} \frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{X_1 \in C_i})}{V(x)} = 0 \leq \rho,$$

so that

$$\sup_{x \in C_i} \mathbb{E}_x\left(V(X_n)\mathbb{1}_{T_{\{\partial\}\cup D\setminus C_i} > n}\right) \leq \rho^n V(x).$$

Therefore, the assumptions of Theorem 4.1 are satisfied by the process $X$ and the finite partition of $D$ given by $\cup_{j \in J} C_j$ and $C_i$, $i \in J'$, so we deduce that $X$ satisfies Assumption (A).

In order to prove the last sentence of Theorem 5.1, we apply the above proof to the process $X$ restricted to $D_{x_0} \cup \{\partial\}$, where

$$D_{x_0} = \{x \in D \text{ such that } \mathbb{P}_{x_0}(\exists n \geq 0,\ X_n = x) > 0\}.$$

We deduce from (4.6) in Theorem 4.1 that $\eta_S(x) > 0$ in Assumption (A) for this process, so that Proposition 2.5(iii) entails the claim for $X$ restricted to $D_{x_0} \cup \{\partial\}$. But the definition of $D_{x_0}$ clearly implies that $T_{\{\partial\}\cup D\setminus D_{x_0}} = \tau_\partial$ $\mathbb{P}_{x_0}$-a.s., which concludes the proof. □

# A Polynomial decay for linear operators on Banach spaces

## A.1 Polynomial decay and first property

Let $\mathfrak{P}$ be a bounded linear operator on a normed real vector space $(B, |\cdot|)$. We consider the following assumption.

**Assumption (H).** There exists a bounded linear operator $E_{\mathfrak{P}}$ on $B$ and $\mathfrak{J}_{\mathfrak{P}} \in \mathbb{R}_+$ such that, for all $x \in B$ and all $n \geq 1$,

$$\left| n^{-\mathfrak{J}_{\mathfrak{P}}} \mathfrak{P}^n x - E_{\mathfrak{P}} x \right| \leq \alpha_{\mathfrak{P},n} |x|, \tag{A.1}$$

where $(\alpha_{\mathfrak{P},n})_{n\in\mathbb{N}}$ is a numerical sequence which converges to 0 when $n \to +\infty$.

We also set $\alpha_{\mathfrak{P},0} = 1$.

*Remark* 22. One natural generalization of Assumption (H) is to consider the situation were $\alpha_{\mathfrak{P},n} = \alpha_{\mathfrak{P},n}(x)$ is allowed to depend on $x$. One easily checks that the following result and its proof remain valid in this setting (replacing $\alpha_{\mathfrak{P},n}$ by $\alpha_{\mathfrak{P},n}(x)$ in (i)). △

**Proposition A.1.** *Assume that (H) holds. Then,*

*(i) for all $n \geq 1$ and all $\ell \geq 0$,*

$$|||\mathfrak{P}^n||| \leq (\alpha_{\mathfrak{P},n} + |||E_{\mathfrak{P}}|||)\, n^{\mathfrak{I}_{\mathfrak{P}}};$$

*(ii) we have $\mathfrak{P}E_{\mathfrak{P}} = E_{\mathfrak{P}}\mathfrak{P} = E_{\mathfrak{P}}$; in particular, $E_{\mathfrak{P}}$ takes its values in the vector space of eigenvectors of $\mathfrak{P}$ associated to $1$;*

*(iii) if $x$ is an eigenvector of $\mathfrak{P}$ associated to $1$, then $E_{\mathfrak{P}}x = \mathbb{1}_{\mathfrak{I}_{\mathfrak{P}}=0}x$.*

*Proof.* The first assertion is immediate. For the second one, we observe that

$$|||\mathfrak{P}E_{\mathfrak{P}} - E_{\mathfrak{P}}||| \leq |||n^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - \mathfrak{P}E_{\mathfrak{P}}||| \\ + |||(n+1)^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - n^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n+1}||| + |||(n+1)^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - E_{\mathfrak{P}}|||$$

where

$$|||n^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - \mathfrak{P}E_{\mathfrak{P}}||| = |||\mathfrak{P}(n^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n} - E_{\mathfrak{P}})||| \leq |||\mathfrak{P}|||\, \alpha_{\mathfrak{P},n}$$

and

$$|||(n+1)^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - n^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n+1}||| \leq \left(\frac{(n+1)^{\mathfrak{I}_{\mathfrak{P}}}}{n^{\mathfrak{I}_{\mathfrak{P}}}} - 1\right)$$

and

$$|||(n+1)^{\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - E_{\mathfrak{P}}||| \leq \alpha_{\mathfrak{P},n+1},$$

so that $|||\mathfrak{P}E_{\mathfrak{P}} - E_{\mathfrak{P}}||| \to 0$ when $n \to +\infty$, and hence

$$\mathfrak{P}E_{\mathfrak{P}} = E_{\mathfrak{P}}.$$

Similarly, for the third assertion, we have

$$|||E_{\mathfrak{P}}\mathfrak{P} - E_{\mathfrak{P}}||| \leq |||n^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - E_{\mathfrak{P}}\mathfrak{P}||| \\ + |||(n+1)^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - n^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n+1}||| + |||(n+1)^{\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - E_{\mathfrak{P}}|||,$$

where

$$|||n^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - E_{\mathfrak{P}}\mathfrak{P}||| \leq |||\mathfrak{P}|||\,|||n^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n} - E_{\mathfrak{P}}||| \leq \alpha_{\mathfrak{P},n},$$

and the other terms go to 0 as in the previous case. Hence, we deduce that

$$E_{\mathfrak{P}}\mathfrak{P} = E_{\mathfrak{P}}.$$

The last assertion is immediate. □

## A.2 Polynomial decay for upper triangular matrix of linear operators

Let $B_1$ and $B_2$ be two Banach spaces, and $\mathfrak{P} : B_1 \to B_1$, $\mathfrak{Q} : B_1 \to B_2$, $\mathfrak{R} : B_2 \to B_2$ three bounded operators. We define the Banach space $B$ as the direct sum of $B_1$ and $B_2$ and consider the operator $\mathfrak{S} = \mathfrak{P} + \mathfrak{Q} + \mathfrak{R}$ on $B$, where $\mathfrak{P}_{|B_2} = \mathfrak{Q}_{|B_2} = \mathfrak{R}_{|B_1} = 0$. Formally, $\mathfrak{S}$ can be represented as the following upper triangular matrix of linear operators:

$$\mathfrak{S} = \begin{bmatrix} \mathfrak{P} & \mathfrak{Q} \\ 0 & \mathfrak{R} \end{bmatrix},$$

so that

$$\mathfrak{S}^n = \begin{bmatrix} \mathfrak{P}^n & \sum_{k=1}^{n} \mathfrak{R}^{n-k}\mathfrak{Q}\mathfrak{P}^{k-1} \\ 0 & \mathfrak{R}^n \end{bmatrix}.$$

The study of the spectrum of such upper triangular matrices of linear operators over a Banach space has already been considered in the literature, see for instance [3, 6, 2, 30] and references therein. In the following propositions, we are interested in the polynomial decay of the operator $\mathfrak{S}$, which is related to the algebraic multiplicity of its leading eigenvalue.

*Remark* 23. Adapting the results of this section and their proofs to the situation where Assumption (H) allows $\alpha_{\mathfrak{P},n}$ to depend on $x$ is straightforward, but leads in some places to additional technical assumptions and cumbersome expressions. Since the situation where $\alpha_{\mathfrak{P},n}$ does not depend on $x$ is sufficiently rich to illustrate the implications of the results below, we decided to only expose this simplest situation. We leave the interested reader to check that if $\alpha_{\mathfrak{P},n} = \alpha_{\mathfrak{P},n}(x)$ is allowed to depend on $x$, then

- Proposition A.2 still holds true, replacing $\alpha_{\mathfrak{P},\ell}$ by $\alpha_{\mathfrak{P},\ell}(x)$ in the expression of $\alpha_{\mathfrak{S},n}$;

- Proposition A.3 still holds true, assuming in addition that

$$\beta_n(x) := \sum_{k=0}^{n-1} \alpha_{\mathfrak{R},n-k-1}(\mathfrak{Q}\mathfrak{P}^k x)\theta_k \xrightarrow[n\to+\infty]{} 0$$

  and replacing $\sum_{k=0}^{n-1} \alpha_{\mathfrak{R},n-k-1}\theta_k$ (resp. $\alpha_{\mathfrak{R},n}$) by $\beta_n(x)$ (resp. $\alpha_{\mathfrak{R},n}(x)$) in the expression of $\alpha_{\mathfrak{S},n}$;

- Proposition A.4 still holds true, assuming in addition that

$$\beta'_n(x) := \frac{1}{n}\sum_{k=1}^{n} \alpha_{\mathfrak{R},n-k}(\mathfrak{Q}\mathfrak{P}_{k-1})(x)\left(\frac{n-k}{n}\right)^{\mathfrak{J}_{\mathfrak{R}}} \xrightarrow[n\to+\infty]{} 0.$$

  and replacing $\sum_{k=1}^{n} \alpha_{\mathfrak{R},n-k}\left(\frac{n-k}{n}\right)^{\mathfrak{J}_{\mathfrak{R}}}$ (resp. $\alpha_{\mathfrak{P},k}$) by $\beta'_n(x)$ (resp. $\alpha_{\mathfrak{P},k}(x)$) in the expression of $\alpha_{\mathfrak{S},n}$.

△

For all $n \geq 0$, we set $\gamma_n = |||\mathfrak{R}^n|||$, $\Gamma_n = \sum_{k\geq n}\gamma_k$, $\theta_n = |||\mathfrak{P}^n|||$ and $\Theta_n = \sum_{k\geq n}\theta_k$. We consider the case $\Gamma_0 < +\infty$ in Proposition A.2, the case $\Theta_0 < +\infty$ in Proposition A.3, and the case $\Gamma_0 = \Theta_0 = +\infty$ in Proposition A.4.

**Proposition A.2.** *Assume that $\sum_{n=0}^{\infty} |||\mathfrak{R}^n||| < +\infty$ and that the operator $\mathfrak{P}$ satisfies Assumption (H). Then $\mathfrak{S}$ satisfies assumption (H) with*

$$\mathfrak{J}_{\mathfrak{S}} = \mathfrak{J}_{\mathfrak{P}} \text{ and } E_{\mathfrak{S}} = E_{\mathfrak{P}} + \sum_{\ell\geq 0} \mathfrak{R}^\ell \mathfrak{Q} E_{\mathfrak{P}},$$

*and*

$$\alpha_{\mathfrak{S},n} = \alpha_{\mathfrak{P},n} + C\Gamma_n + C\sum_{k=0}^{n-1}\gamma_k\left(\frac{(\alpha_{\mathfrak{P},n-k-1}+1)\mathfrak{J}_{\mathfrak{P}}(k+1)}{n} + \alpha_{\mathfrak{P},n-k-1}\right),$$

*for some positive constant $C > 0$ (which depends on $|||\mathfrak{Q}|||$ and $|||E_{\mathfrak{P}}|||$) and with the convention that $\alpha_{\mathfrak{P},0} = 1$.*

*Proof.* Fix $n \geq 1$ and $x \in B_1$. Then

$$|n^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{S}^n x - E_{\mathfrak{S}}x| \leq |n^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^n x - E_{\mathfrak{P}}x| + \sum_{k=0}^{n-1}\left|n^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{R}^k\mathfrak{Q}\mathfrak{P}^{n-k-1}x - \mathfrak{R}^k\mathfrak{Q}E_{\mathfrak{P}}x\right| + \sum_{k=n}^{\infty}|\mathfrak{R}^k\mathfrak{Q}E_{\mathfrak{P}}x|. \quad \text{(A.2)}$$

By Assumption (H), the first term on the right hand side is bounded by $\alpha_{\mathfrak{P},n}(x)|x|$. Moreover, since $\mathfrak{Q}$ and $E_{\mathfrak{P}}$ are bounded operators, $|\mathfrak{R}^k\mathfrak{Q}E_{\mathfrak{P}}x| \le \|\!|\mathfrak{Q}E_{\mathfrak{P}}\|\!|\gamma_k|x|$ and hence

$$|n^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^n x - E_{\mathfrak{P}}x| + \left|\sum_{k=n}^{\infty}\mathfrak{R}^k\mathfrak{Q}E_{\mathfrak{P}}x\right| \le \alpha_{\mathfrak{P},n}|x| + \|\!|\mathfrak{Q}E_{\mathfrak{P}}\|\!|\Gamma_n|x|.$$

For the second term in the r.h.s. of (A.2), we have for all $k \in \{0,\dots,n-2\}$,

$$\begin{aligned}\left|n^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{R}^k\mathfrak{Q}\mathfrak{P}^{n-k-1}x - \mathfrak{R}^k\mathfrak{Q}E_{\mathfrak{P}}x\right| &\le \left|1-\left(\frac{n-k-1}{n}\right)^{\mathfrak{I}_{\mathfrak{P}}}\right| \|\!|\mathfrak{R}^k\mathfrak{Q}\|\!| \left|(n-k-1)^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n-k-1}x\right| \\ &\quad + \|\!|\mathfrak{R}^k\mathfrak{Q}\|\!| \left|(n-k-1)^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{P}^{n-k-1}x - E_{\mathfrak{P}}x\right| \\ &\le \|\!|\mathfrak{Q}\|\!| \frac{\mathfrak{I}_{\mathfrak{P}}(k+1)}{n}\gamma_k\left(\alpha_{\mathfrak{P},n-k-1} + \|\!|E_{\mathfrak{P}}\|\!|\right)|x| \\ &\quad + \|\!|\mathfrak{Q}\|\!|\gamma_k\alpha_{\mathfrak{P},n-k-1}|x|,\end{aligned}$$

where we used Proposition A.1 (i) and Assumption (H). For $k = n-1$, we observe that

$$\left|n^{-\mathfrak{I}_{\mathfrak{P}}}\mathfrak{R}^k\mathfrak{Q}\mathfrak{P}^{n-k-1}x - \mathfrak{R}^k\mathfrak{Q}E_{\mathfrak{P}}x\right| \le 2(\|\!|\mathfrak{Q}\|\!| + \|\!|\mathfrak{Q}E_{\mathfrak{P}}\|\!|)\gamma_{n-1}|x|.$$

If $x \in B_2$, then $\mathfrak{S}^n x = \mathfrak{R}^n x$ and $E_{\mathfrak{P}}x = 0$, so that $|\mathfrak{S}^n x - E_{\mathfrak{S}}x| \le \gamma_n|x| \le \Gamma_n|x|$.

The above bounds and (A.2) show that $\mathfrak{S}$ satisfies Assumption (H) with

$$\alpha_{\mathfrak{S},n} = \alpha_{\mathfrak{P},n} + C\Gamma_n + C\sum_{k=0}^{n-1}\gamma_k\left(\frac{(\alpha_{\mathfrak{P},n-k-1}+1)\mathfrak{I}_{\mathfrak{P}}(k+1)}{n} + \alpha_{\mathfrak{P},n-k-1}\right),$$

which converges to 0 when $n \to +\infty$.

□

**Proposition A.3.** *Assume that $\sum_{n=0}^{\infty}\|\!|\mathfrak{P}^n\|\!| < +\infty$ and that the operator $\mathfrak{R}$ satisfies Assumption (H). Then $\mathfrak{S}$ satisfies assumption (H) with*

$$\mathfrak{I}_{\mathfrak{S}} = \mathfrak{I}_{\mathfrak{R}} \text{ and } E_{\mathfrak{S}} = E_{\mathfrak{R}} + \sum_{\ell\ge 0}E_{\mathfrak{R}}\mathfrak{Q}\mathfrak{P}^{\ell},$$

*and*

$$\alpha_{\mathfrak{S},n} = \alpha_{\mathfrak{R},n} + C\Theta_n + C\sum_{k=0}^{n-1}\theta_k\left(\alpha_{\mathfrak{R},n-k-1} + \frac{\mathfrak{I}_{\mathfrak{S}}\,k}{n}\right),$$

*for some positive constant $C > 0$ (which depends on $\|\!|\mathfrak{Q}\|\!|$ and $\|\!|E_{\mathfrak{R}}\mathfrak{Q}\|\!|$) and with the convention that $\alpha_{\mathfrak{R},0} = 1$.*

*Proof.* We have, for all $n \geq 1$ and all $x \in B$,

$$\left|n^{-\mathfrak{J}_{\mathfrak{S}}}\mathfrak{S}^n x - E_{\mathfrak{S}}x\right| \leq \left|n^{-\mathfrak{J}_{\mathfrak{R}}}\mathfrak{R}^n x - E_{\mathfrak{R}}x\right| + n^{-\mathfrak{J}_{\mathfrak{R}}}\left|\mathfrak{P}^n x\right| + n^{-\mathfrak{J}_{\mathfrak{R}}}\sum_{k=0}^{n-1}\left|\mathfrak{R}^{n-k-1}\mathfrak{Q}\mathfrak{P}^k x - (n-k-1)^{\mathfrak{J}_{\mathfrak{R}}}E_{\mathfrak{R}}\mathfrak{Q}\mathfrak{P}^k x\right| + \sum_{k=0}^{n-1}\left(1-\left(\frac{n-k-1}{n}\right)^{\mathfrak{J}_{\mathfrak{R}}}\right)|E_{\mathfrak{R}}\mathfrak{Q}\mathfrak{P}^k x| + \sum_{k=n}^{\infty}|E_{\mathfrak{R}}\mathfrak{Q}\mathfrak{P}^k x|.$$

Using Assumption (H) for $\mathfrak{R}$ and the fact that $\mathfrak{Q}$ is a bounded operator, we deduce that the first three terms are bounded by

$$\alpha_{\mathfrak{R},n}|x| + \theta_n|x| + |||\mathfrak{Q}|||\sum_{k=0}^{n-1}\alpha_{\mathfrak{R},n-k-1}\left(\frac{n-k-1}{n}\right)^{\mathfrak{J}_R}\theta_k|x| \leq \left(\alpha_{\mathfrak{R},n} + |||\mathfrak{Q}|||\sum_{k=0}^{n-1}\alpha_{\mathfrak{R},n-k-1}\theta_k\right)|x|.$$

The fourth and fifth terms are bounded by

$$\sum_{k=0}^{n-1}|||E_{\mathfrak{R}}\mathfrak{Q}|||\left(1-\left(\frac{n-k-1}{n}\right)^{\mathfrak{J}_{\mathfrak{R}}}\right)\theta_k|x| + \sum_{k=n}^{\infty}|||E_{\mathfrak{R}}\mathfrak{Q}|||\theta_k|x| \leq |||E_{\mathfrak{R}}\mathfrak{Q}|||\left(\sum_{k=0}^{n-1}\frac{\mathfrak{J}_{\mathfrak{R}}k}{n}\theta_k + \Theta_n\right)|x|.$$

We finally deduce that

$$\left|n^{-j_{\mathfrak{S}}}\mathfrak{S}^n x - E_{\mathfrak{S}}x\right| \leq \alpha_{\mathfrak{S},n}|x|,$$

where

$$\alpha_{\mathfrak{S},n} = \alpha_{\mathfrak{R},n} + |||\mathfrak{Q}|||\sum_{k=0}^{n-1}\alpha_{\mathfrak{R},n-k-1}\theta_k + (|||E_{\mathfrak{R}}\mathfrak{Q}||| + 1)\Theta_n + |||E_{\mathfrak{R}}\mathfrak{Q}|||\sum_{k=0}^{n-1}\frac{\mathfrak{J}_{\mathfrak{S}}k}{n}\theta_k. \quad \square$$

**Proposition A.4.** *Assume $\mathfrak{P}$ and $\mathfrak{R}$ both satisfy Assumption (H), with $\mathfrak{J}_{\mathfrak{P}} = 0$. Then $\mathfrak{S}$ satisfies Assumption (H) with*

$$\mathfrak{J}_{\mathfrak{S}} = 1 + \mathfrak{J}_{\mathfrak{R}} \textit{ and } E_{\mathfrak{S}} = \frac{1}{\mathfrak{J}_{\mathfrak{S}}}E_{\mathfrak{R}}\mathfrak{Q}E_{\mathfrak{P}}$$

*and*

$$\alpha_{\mathfrak{S},n} \quad = \quad \frac{C}{n}\Big(\mathfrak{J}_{\mathfrak{R}} \quad + \quad \sum_{k=0}^{n} \alpha_{\mathfrak{P},k} \quad + \quad \left(\max_{k\geq 0} \alpha_{\mathfrak{P},k}+1\right) \sum_{k=0}^{n} \alpha_{\mathfrak{R},n-k} \left(\frac{n-k}{n}\right)^{\mathfrak{J}_{\mathfrak{R}}} \Big)$$

*for some positive constant $C$ (which depends on $\|\|E_{\mathfrak{R}}\mathfrak{Q}E_{\mathfrak{P}}\|\|$, $\|\|E_{\mathfrak{R}}\mathfrak{Q}\|\|$, $\|\|\mathfrak{Q}\|\|$ and $\|\|E_{\mathfrak{P}}\|\|$), and with the convention that $\alpha_{\mathfrak{P},0} = \alpha_{\mathfrak{R},0} = 1$.*

*Proof.* Using the fact that $\mathfrak{S}^n x = \mathfrak{P}^n x + \sum_{k=1}^{n} \mathfrak{R}^{n-k}\mathfrak{Q}\mathfrak{P}^{k-1}x$, we deduce that

$$\begin{aligned}
\left|n^{-\mathfrak{J}_{\mathfrak{S}}}\mathfrak{S}^n x - E_{\mathfrak{S}}x\right| \leq n^{-\mathfrak{J}_{\mathfrak{S}}}\mathfrak{P}^n x + n^{-\mathfrak{J}_{\mathfrak{S}}}\mathfrak{R}^n x \\
&+ n^{-\mathfrak{J}_{\mathfrak{S}}} \sum_{k=1}^{n} \left|\mathfrak{R}^{n-k}\mathfrak{Q}\mathfrak{P}^{k-1}x - (n-k)^{\mathfrak{J}_{\mathfrak{R}}} E_{\mathfrak{R}}\mathfrak{Q}\mathfrak{P}^{k-1}x\right| \\
&+ n^{-\mathfrak{J}_{\mathfrak{S}}} \sum_{k=1}^{n} (n-k)^{\mathfrak{J}_{\mathfrak{R}}} \left|E_{\mathfrak{R}}\mathfrak{Q}\mathfrak{P}^{k-1}x - E_{\mathfrak{R}}\mathfrak{Q}E_{\mathfrak{P}}x\right| \\
&+ \left|n^{-\mathfrak{J}_{\mathfrak{S}}} \sum_{k=1}^{n} (n-k)^{\mathfrak{J}_{\mathfrak{R}}} - \frac{1}{\mathfrak{J}_{\mathfrak{S}}}\right| \left|E_{\mathfrak{R}}\mathfrak{Q}E_{\mathfrak{P}}x\right| \qquad \text{(A.3)}
\end{aligned}$$

For the first two terms on the right hand side, we deduce from Proposition A.1 (i) that

$$n^{-\mathfrak{J}_{\mathfrak{S}}}|\mathfrak{P}^n x| + n^{-\mathfrak{J}_{\mathfrak{S}}}\mathfrak{R}^n x \leq (\alpha_{\mathfrak{P},n} + \alpha_{\mathfrak{R},n} + \|\|E_{\mathfrak{P}}\|\| + \|\|E_{\mathfrak{R}}\|\|)\, n^{-1}\, |x|. \qquad \text{(A.4)}$$

For the third term, we use that, for all $n \geq 1$, Proposition A.1 (i) and the boundedness of $\mathfrak{Q}$ imply

$$|\mathfrak{Q}\mathfrak{P}^{n-1}x| \leq \|\|\mathfrak{Q}\|\|(\alpha_{\mathfrak{P},n-1} + \|\|E_{\mathfrak{P}}\|\|)|x|.$$

Hence, using Assumption (H) for $\mathfrak{R}$, we obtain, for all $k \geq 1$,

$$\left|\mathfrak{R}^{n-k}\mathfrak{Q}\mathfrak{P}^{k-1}x - (n-k)^{\mathfrak{J}_{\mathfrak{R}}} E_{\mathfrak{R}}\mathfrak{Q}\mathfrak{P}^{k-1}x\right| \leq \alpha_{\mathfrak{R},n-k}(n-k)^{\mathfrak{J}_{\mathfrak{R}}}\|\|\mathfrak{Q}\|\|(\alpha_{\mathfrak{P},k-1} + \|\|E_{\mathfrak{P}}\|\|)|x|.$$

Thus

$$\begin{aligned}
n^{-\mathfrak{J}_{\mathfrak{S}}} \sum_{k=1}^{n} &\left|\mathfrak{R}^{n-k}\mathfrak{Q}\mathfrak{P}^{k-1}x - (n-k)^{\mathfrak{J}_{\mathfrak{R}}} E_{\mathfrak{R}}\mathfrak{Q}\mathfrak{P}^{k-1}x\right| \\
&\leq \|\|\mathfrak{Q}\|\|\,|x| \left(\max_{k\geq 0} \alpha_{\mathfrak{P},k} + \|\|E_{\mathfrak{P}}\|\|\right) \frac{1}{n} \sum_{k=1}^{n} \alpha_{\mathfrak{R},n-k} \left(\frac{n-k}{n}\right)^{\mathfrak{J}_{\mathfrak{R}}}. \qquad \text{(A.5)}
\end{aligned}$$

For the fourth term, we use Assumption (H) for $\mathfrak{P}$ to derive

$$n^{-\mathfrak{I}_{\mathfrak{S}}} \sum_{k=1}^{n} (n-k)^{\mathfrak{I}_{\mathfrak{R}}} \left| E_{\mathfrak{R}} \mathfrak{Q} \mathfrak{P}^{k-1} x - E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}} x \right| \leq \frac{|||E_{\mathfrak{R}} \mathfrak{Q}||| \, |x|}{n} \sum_{k=1}^{n} \alpha_{\mathfrak{P},k-1} (1-k/n)^{\mathfrak{I}_{\mathfrak{R}}}. \tag{A.6}$$

Finally, the fifth term in (A.3) is bounded by

$$|||E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}}||| \left| n^{-\mathfrak{I}_{\mathfrak{S}}} \sum_{k=1}^{n} (n-k)^{\mathfrak{I}_{\mathfrak{R}}} - \frac{1}{\mathfrak{I}_{\mathfrak{S}}} \right| |x| \tag{A.7}$$

$$\leq |||E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}}||| \sum_{k=1}^{n} \left| \frac{1}{n} (1-k/n)^{\mathfrak{I}_{\mathfrak{R}}} - \int_{(k-1)/n}^{k/n} (1-u)^{\mathfrak{I}_{\mathfrak{R}}} \, du \right| |x|$$

$$\leq |||E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}}||| \frac{\mathfrak{I}_{\mathfrak{R}}}{n} |x|. \tag{A.8}$$

Combining (A.3) and the bounds (A.4), (A.5), (A.6), (A.8) ends the proof of Proposition A.4. □

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