

Transcritical bifurcation for the conditional distribution of a diffusion process

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Abstract

In this article, we describe a simple class of models of absorbed diffusion processes with parameter, whose conditional law exhibits a transcritical bifurcation. Our proofs are based on the description of the set of quasi-stationary distributions for general two-clusters reducible processes.

1 Model, motivation and main result

Let $D = (0, 5)$ and consider Lipschitz functions $\varphi^1, \varphi^2, \psi^1, \psi^2 : D \rightarrow [0, 1]$ such that $\psi_1 + \psi_2 = 1$ and (see Figure 1)

$$\begin{cases} \varphi_{(0,1]}^1 \equiv 1, \varphi_{[1,2]}^1 \leq 1, \varphi_{[2,5]}^1 \equiv 0, \\ \varphi_{(0,3]}^2 \equiv 0, \varphi_{[3,4]}^2 \leq 1, \varphi_{[4,5]}^2 \equiv 1, \\ \psi_{(0,2]}^1 \equiv 1, 0 < \psi_{(2,3]}^1 \leq 1, \psi_{[3,5]}^1 \equiv 0, \\ \psi_{(0,2]}^2 \equiv 0, 0 < \psi_{(2,3]}^2 \leq 1, \psi_{[3,5]}^2 \equiv 1. \end{cases}$$

For all $\alpha > 0$, we consider the absorbed diffusion process X^α evolving according to the Itô's stochastic differential equation

$$dX_t^\alpha = (\varphi^1(X_t^\alpha) + \sqrt{\alpha} \varphi^2(X_t^\alpha)) dB_t + (\psi^1(X_t^\alpha) + \alpha \psi^2(X_t^\alpha)) dt, \quad (1)$$

stopped when it reaches $\partial D = \{0, 5\}$, and where B is a standard one dimensional Brownian motion. This defines a sub-Markov semi-group $(P_t^\alpha)_{t \in \mathbb{R}_+}$ on D by

$$\delta_x P_t^\alpha f = \mathbb{E}_x(f(X_t^\alpha) 1_{X_t^\alpha \in D}), \quad \forall f \in L^\infty(D)$$

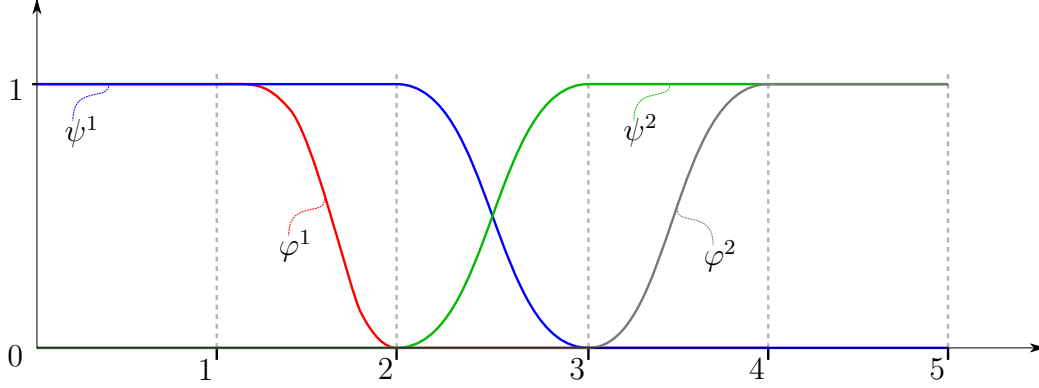


Figure 1: Representation of the graphs of $\varphi^1, \varphi^2, \psi^1$ and ψ^2 .

and a semi-flow $(\Phi_t^\alpha)_{t \in \mathbb{R}_+}$ on the set $\mathcal{P}(D)$ of probability measures on D by

$$\Phi_t^\alpha(\mu) = \mathbb{P}_\mu(X_t \in \cdot \mid X_t \in D).$$

Observe that the diffusion coefficient in (1) vanishes on the interval $[2, 3]$. Since in addition the drift coefficient is positive on $[2, 3]$ for any $\alpha > 0$, the set $[3, 5]$ is absorbing and $\mathbb{P}_2(\forall t \geq 0, X_t \geq 2, \exists t \geq 0, X_t = 3) = 1$.

Hence, the family of diffusion processes $(X^\alpha)_{\alpha \geq 0}$ has some similarities with the family of discrete-time Markov chains $(X^{a,b})_{a,b \in (0,1)}$, where $X^{a,b}$ is defined on $\{1, 2, \partial\}$, absorbed at ∂ , with transition submatrix on $\{1, 2\}$ given by

$$\begin{pmatrix} a & 1-a \\ 0 & b \end{pmatrix}.$$

It is pointed in [3, Example 3.5] that the probability measures $\nu_2 := \delta_2$ and $\nu := \frac{a-b}{1-b}\delta_1 + \frac{1-a}{1-b}\delta_2$ (when $a > b$) are such that

- If $a > b$, $\lim_{n \rightarrow \infty} \mathbb{P}_\mu(X_n^{a,b} = i \mid X_n^{a,b} \neq \partial) = \nu(\{i\})$ for all $i = 1, 2$ and $\mu \neq \delta_2$.
- Otherwise, $\lim_{n \rightarrow \infty} \mathbb{P}_\mu(X_n^{a,b} = i \mid X_n^{a,b} \neq \partial) = \nu_2(\{i\})$ for all $i = 1, 2$ and $\mu \in \mathcal{P}(\{1, 2\})$.

Our goal in this note is to prove a similar property for the family $(X^\alpha)_{\alpha \geq 0}$, which can be formulated as a bifurcation of the dynamical system on $\mathcal{P}(D)$ generated by Φ^α . Our motivation is to quantify precisely the speed of convergence and the basin of attraction of the fixed points of this dynamical

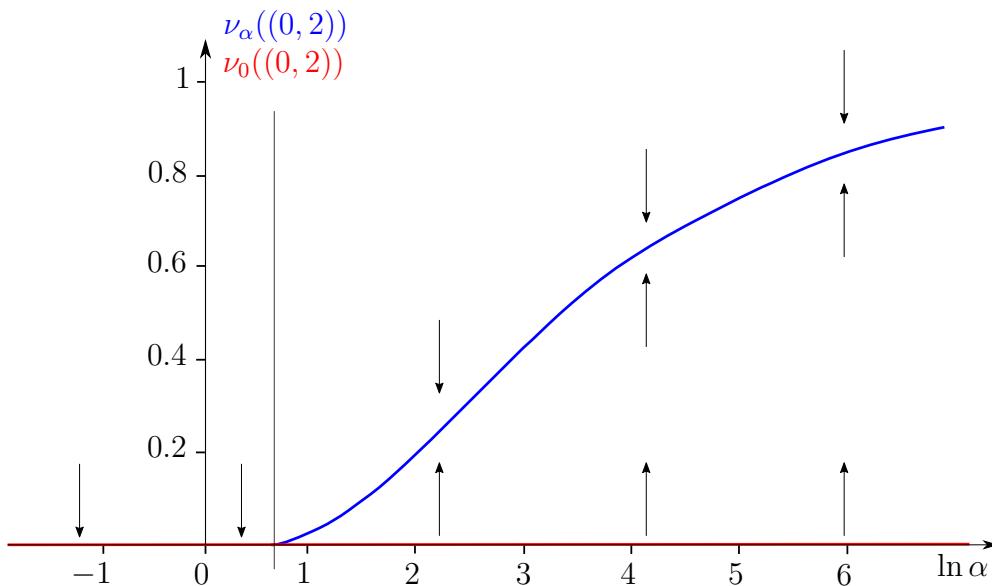


Figure 2: Domain of attraction of the attractors, with quadratic $\varphi^1, \varphi^2, \psi^1, \psi^2$. $\nu_\alpha((0, 2))$ has been computed numerically using Fleming-Viot type approximation techniques (see for instance [15]).

systems (see Fig. 2). Another motivation is to provide an illustration to the fact that, for an absorbed diffusion process satisfying the weak Hörmander condition and regularity properties at the absorbing set, uniqueness of a quasi-stationary distribution does not necessary hold true unless some accessibility properties are satisfied (see Theorem 1.8 of [2]). The definition of a quasi-stationary distribution is recalled in Section 2 below.

To state our main result, we define the absorption parameters of X^α for $\alpha = 1$:

$$\lambda_1 := \inf \left\{ \lambda \in \mathbb{R}, \liminf_{t \rightarrow +\infty} e^{\lambda t} \mathbb{P}_{x_1}(X_t^1 \in (0, 2)) > 0 \right\}, \text{ for some } x_1 \in (0, 2).$$

and

$$\lambda_2 := \inf \left\{ \lambda \in \mathbb{R}, \liminf_{t \rightarrow +\infty} e^{\lambda t} \mathbb{P}_{x_2}(X_t^1 \in [3, 5]) > 0 \right\}, \text{ for some } x_2 \in [3, 5],$$

As expected, we will see that the parameters λ_1 and λ_2 are positive and do not depend on x_1 nor x_2 .

Theorem 1.1. *The dynamical system generated by Φ^α parametrized by $\alpha > 0$ admits a transcritical bifurcation at λ_1/λ_2 . More precisely, there exist a family of probability measure $(\mu_\alpha)_{\alpha > \lambda_1/\lambda_2}$ on $(0, 5)$ and a probability measure μ_0 on $[3, 5)$ such that:*

- *for $\alpha \leq \lambda_1/\lambda_2$, μ_0 is a global attractor for Φ^α for the total variation distance,*
- *for $\alpha > \lambda_1/\lambda_2$, μ_0 is a saddle point whose stable manifold for the total variation distance is the set $W^s(\mu_0) := \{\mu \in \mathcal{P}((0, 5)) : \mu((0, 2)) = 0\}$, and μ_α is a stable point whose basin of attraction for the total variation distance is $W^u(\mu_0) = \{\mu \in \mathcal{P}((0, 5)) : \mu((0, 2)) > 0\}$.*

Our method also provides an estimate for the speed of convergence of the dynamical system generated by Φ^α to its limit fixed point. One can check from the proof that it is exponential when $\alpha \neq \lambda_1/\lambda_2$ and polynomial in $O(1/t)$ when $\alpha = \lambda_1/\lambda_2$.

The proof of this result relies on the theory of quasi-stationary distributions. We show in particular that the process X^α admits either one or two quasi-stationary distributions, depending on the value of α , which correspond to μ_0 and μ_α (when $\alpha > \lambda_0/\lambda_1$). A central feature allowing this property is that X^α is reducible. Indeed, it is proved in [2] that irreducibility for such diffusions entails the uniqueness of a quasi-stationary distribution. In order to prove Theorem 1.1, we start with considerations on quasi-stationary distributions for Markov processes in reducible state spaces. Since they apply to general Markov processes and may have independent interest, they are stated independently in Section 2. We conclude the proof of the theorem in Section 3.

2 Quasi-stationarity for two-clusters reducible processes

Let X be a Markov process with state space $M = D \cup \{\partial\}$ with $\partial \notin D$, in discrete or continuous time, such that M admits a measurable partition $D_1 \cup D_2 \cup \{\partial\}$. We assume that $\{\partial\}$ and $D_2 \cup \{\partial\}$ are absorbing sets (see Figure 3), which means that

$$\mathbb{P}_\partial(X_t = \partial) = 1 \text{ and } \mathbb{P}_x(X_t \in D_2 \cup \{\partial\}) = 1, \forall x \in D_2, \forall t \geq 0.$$

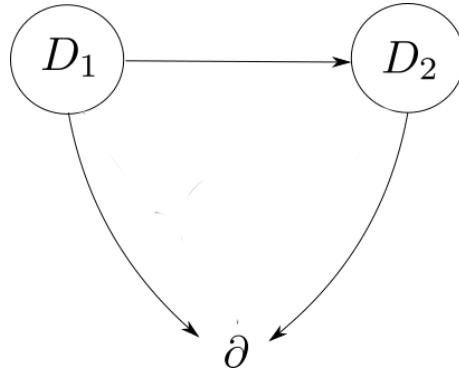


Figure 3: Transition graph displaying the relation between the sets D_1 , D_2 and ∂ .

A probability measure ν is said to be a *quasi-stationary distribution* if, for all $t \geq 0$,

$$\mathbb{P}_\nu(X_t \in \cdot | X_t \neq \partial) = \nu(\cdot). \quad (2)$$

It is well-known (see [24, Proposition 1]) that the notion of quasi-stationary distribution is equivalent to the one of *quasi-limiting distribution*, defined as a probability measure ν such that there exist some initial distributions μ such that, for all measurable subset $A \subset D$,

$$\lim_{t \rightarrow \infty} \mathbb{P}_\mu(X_t \in A | X_t \neq \partial) = \nu(A).$$

It is also well-known that, to any quasi-stationary distribution ν is associated the so-called *exponential absorption rate* $\lambda_0 > 0$ such that $\mathbb{P}_\nu(X_t \in \cdot, X_t \neq \partial) = e^{-\lambda_0 t} \nu$. We refer the reader to [24, 12, 27] for a general overview on the theory of quasi-stationarity, and to [25, 17, 5, 6, 20, 21, 23, 15, 8, 7, 19, 18, 22, 26] for the study of the quasi-stationary distribution of diffusion processes.

The aim of this section is to provide conditions on X allowing to obtain the existence of a quasi-stationary distribution ν for the process X , as well as the so-called *Malthusian behavior* (see [4] for the terminology), that is to say the existence of a positive function η on D such that

$$\lim_{t \rightarrow \infty} e^{\lambda_0 t} \mathbb{P}_x(X_t \in \cdot, X_t \neq \partial) = \eta(x) \nu(\cdot),$$

where λ_0 is the exponential absorption rate associated to ν (in particular, this convergence entails the convergence of the conditional probability measure

$\mathbb{P}_x(X_t \in \cdot | X_t \neq \partial)$ towards ν). In what follows, we will present three different sets of assumptions, each discussed in three different subsections, and each entailing different Malthusian behavior for the process X .

2.1 Exponential convergence on D_2 and faster exit from D_1

Let us introduce our first set of assumptions.

Assumption QSD2.

- a. There exist a positive function η_2 on D_2 , a probability measure ν_2 on D_2 , a constant $\lambda_2 > 0$, and positive constants $C_2, \gamma_2 > 0$ such that

$$\left\| e^{\lambda_2 t} \mathbb{P}_x(X_t \in \cdot, X_t \in D_2) - \eta_2(x) \nu_2(\cdot) \right\|_{TV} \leq C_2 e^{-\gamma_2 t}, \quad \forall x \in D_2, t \geq 0. \quad (3)$$

- b. In addition, $\sup_{x \in D_1} e^{\lambda_2 t} \mathbb{P}_x(X_t \in D_1) \leq f(t)$, where f is non-increasing and L^1 on \mathbb{R}_+ .

The assumption a. refers to the Malthusian behavior, as described before, of the restriction of X on the subset D_2 , holding uniformly in x in total variation and exponentially fast. We refer the reader to [14, 13, 9, 16, 28, 1, 10, 18, 2, 11] for general criteria entailing such behavior.

Also, remark that the inequality (3) entails that η_2 is bounded (take for example $t = 0$). In addition, it implies that $\eta_2(x) = \lim_{t \rightarrow +\infty} e^{\lambda_2 t} \mathbb{P}_x(X_t \neq \partial)$ for all $x \in D_2$. In addition, as noticed above, it implies that ν_2 is a quasi-stationary distribution for X .

Then Assumption QSD2 entails the following result on quasi-stationarity.

Theorem 2.1. *Under QSD2, there exists $\eta : D \rightarrow \mathbb{R}_+$ positive on D_2 such that*

$$\sup_{x \in D} \left\| e^{\lambda_2 t} \mathbb{P}_x(X_t \in \cdot, X_t \neq \partial) - \eta(x) \nu_2(\cdot \cap D_2) \right\|_{TV} \xrightarrow{t \rightarrow +\infty} 0. \quad (4)$$

If in addition $\mathbb{P}_x(\exists n \geq 0, X_n \in D_2) > 0$ for all $x \in D_1$, then η is positive on D and $\nu_2(\cdot \cap D_2)$ is the unique quasi-stationary distribution for X on D .

In other terms, the Malthusian behavior of X , only assumed for $x \in D_2$ in Assumption QSD2, holds for all $x \in D$ and uniformly in x in total variation. Speed of convergence for (4) is discussed after the proof of Theorem 2.1.

Proof of Theorem 2.1. For the rest, we first prove this theorem in the discrete time setting, and then consider the continuous-time setting.

Step 1: Proof in the discrete-time setting. We define the stopping time $\tau_1^c := \min\{n \geq 0, X_n \notin D_1\}$. For all $x \in D$ and all measurable set $A \subset D_1 \cup D_2$, for all $n \in \mathbb{Z}_+$, we have, using the strong Markov property at time τ_1^c ,

$$\begin{aligned} \mathbb{P}_x(X_n \in A) &= \mathbb{P}_x(X_n \in A \cap D_1) + \mathbb{P}_x(X_n \in A \cap D_2) \\ &= \mathbb{P}_x(X_n \in A \cap D_1) + \sum_{k=0}^n \mathbb{E}_x \left(1_{\tau_1^c=k} \mathbb{P}_{X_k}(X_{n-k} \in A \cap D_2) \right). \end{aligned} \quad (5)$$

If $x \in D_2$, then the result is an immediate consequence of (3) with $\eta(x) = \eta_2(x)$. It only remains to consider the case $x \in D_1$. On the one hand, we have by assumption

$$e^{\lambda_2 n} \mathbb{P}_x(X_n \in A \cap D_1) \leq e^{\lambda_2 n} \mathbb{P}_x(X_n \in D_1) \xrightarrow{n \rightarrow +\infty} 0. \quad (6)$$

On the other hand, using (3) and extending η_2 to $\{\partial\}$ by $\eta_2(\partial) = 0$, we obtain, for all $k \geq 0$, $t \geq 0$ and measurable set $A \subset D_2$,

$$\begin{aligned} &\left| \mathbb{E}_x \left(1_{\tau_1^c=k} \mathbb{P}_{X_k}(X_{n-k} \in A) \right) - \mathbb{E}_x \left(1_{\tau_1^c=k} \eta_2(X_k) \nu_2(A) e^{-\lambda_2(n-k)} \right) \right| \\ &\leq \mathbb{E}_x \left(1_{\tau_1^c=k} C_2 e^{-(\lambda_2 + \gamma_2)(n-k)} 1_{X_k \in D_2} \right) \\ &\leq \begin{cases} C_2 \mathbb{P}_x(X_{k-1} \in D_1, X_k \in D_2) e^{-(\lambda_2 + \gamma_2)(n-k)} & \text{if } k \geq 1, \\ 0 & \text{if } k = 0. \end{cases} \end{aligned}$$

Summing over k , we deduce that, for all $n \geq 0$ and all measurable set $A \subset D_2$,

$$\begin{aligned} &\left| \sum_{k=0}^n \mathbb{E}_x \left(1_{\tau_1^c=k} \mathbb{P}_{X_k}(X_{n-k} \in A) \right) - \sum_{k=0}^n \mathbb{E}_x \left(1_{\tau_1^c=k} \eta_2(X_k) \nu_2(A) e^{-\lambda_2(n-k)} \right) \right| \\ &\leq C_2 e^{-\lambda_2 n} e^{-\gamma_2 n} \sum_{k=1}^n e^{\lambda_2 k} e^{\gamma_2 k} \mathbb{P}_x(X_{k-1} \in D_1, X_k \in D_2) \\ &\leq C_2 e^{-\lambda_2(n-1)} e^{-\gamma_2 n} \sum_{k=1}^n e^{\gamma_2 k} f(k-1). \end{aligned} \quad (7)$$

Moreover

$$\begin{aligned}
\sum_{k \geq n+1} \mathbb{E}_x (1_{\tau_1^c = k} \eta_2(X_k) e^{\lambda_2 k}) &\leq \|\eta_2\|_\infty \sum_{k \geq n+1} e^{\lambda_2 k} \mathbb{P}_x(X_{k-1} \in D_1, X_k \in D_2) \\
&\leq \|\eta_2\|_\infty \sum_{k \geq n+1} e^{\lambda_2 k} (\mathbb{P}_x(X_{k-1} \in D_1) - \mathbb{P}_x(X_k \in D_1)) \\
&\leq \|\eta_2\|_\infty (e^{\lambda_2} - 1) \sum_{k \geq n} e^{\lambda_2 k} \mathbb{P}_x(X_k \in D_1) \\
&\leq \|\eta_2\|_\infty (e^{\lambda_2} - 1) \int_{n-1}^{+\infty} f(t) dt.
\end{aligned}$$

This, (5), (6) and (7) entail that, for all $x \in D_1$ and all measurable $A \subset D$,

$$\sup_{x \in D_1} \left| e^{\lambda_2 n} \mathbb{P}_x(X_n \in A) - \sum_{k=0}^{+\infty} \mathbb{E}_x (1_{\tau_1^c = k} \eta_2(X_k) e^{\lambda_2 k}) \nu_2(A \cap D_2) \right| \xrightarrow{n \rightarrow +\infty} 0.$$

Setting

$$\eta(x) := \begin{cases} \sum_{k=0}^{+\infty} \mathbb{E}_x (1_{\tau_1^c = k} \eta_2(X_k) e^{\lambda_2 k}) & \text{if } x \in D_1 \\ \eta_2(x) & \text{if } x \in D_2, \end{cases}$$

this concludes the proof of (4).

If in addition $\mathbb{P}_x(\exists n \geq 0, X_n \in D_2) > 0$ for all $x \in D_1$, then η is positive on D and hence $\nu_2(\cdot \cap D_2)$ is the unique quasi-limiting distribution of the process and hence its unique quasi-stationary distribution.

Step 2 : Proof in the continuous-time setting. The proof is done by applying the discrete time result to the Markov chain $(X_n)_{n \in \mathbb{Z}_+}$.

Let X satisfy Assumption QSD2. Then the discrete time process $(X_n)_{n \in \mathbb{N}}$ also satisfies Assumption QSD2 and hence, by Theorem 2.1 in the discrete time setting, we have

$$\sup_{x \in D} \left\| e^{\lambda_2 n} \mathbb{P}_x(X_n \in \cdot, X_n \neq \partial) - \eta(x) \nu_2(\cdot \cap D_2) \right\|_{TV} \xrightarrow{n \rightarrow +\infty} 0. \quad (8)$$

For any $x \in D$ and $h > 0$, integrating the above convergence result with respect to $\mathbb{P}_x(X_h \in \cdot)$ entails that

$$\left\| e^{\lambda_2 n} \mathbb{P}_x(X_{n+h} \in \cdot, X_{n+h} \neq \partial) - \mathbb{E}_x(\eta(X_h)) \nu_2(\cdot \cap D_2) \right\|_{TV} \xrightarrow{n \rightarrow +\infty} 0$$

Similarly, for any $x \in D$ and $h > 0$, applying (8) to the test function $y \in D \mapsto \mathbb{P}_y(X_h \in \cdot, X_h \neq \partial)$, gives

$$\|e^{\lambda_2 n} \mathbb{P}_x(X_{n+h} \in \cdot, X_{n+h} \neq \partial) - \eta(x) \mathbb{P}_{\nu_2}(X_h \in \cdot, X_h \neq \partial)\|_{TV} \xrightarrow{n \rightarrow +\infty} 0.$$

But $D_2 \cup \{\partial\}$ being absorbing, we have $\mathbb{P}_{\nu_2}(X_h \in \cdot, X_h \neq \partial) = \mathbb{P}_{\nu_2}(X_h \in \cdot, X_h \in D_2)$, which implies that

$$\mathbb{E}_x(\eta(X_h)) \nu_2(\cdot \cap D_2) = \eta(x) \mathbb{P}_{\nu_2}(X_h \in \cdot \cap D_2) = e^{-\lambda_2 h} \eta(x) \nu_2(\cdot \cap D_2),$$

so that $\mathbb{E}_x(\eta(X_h)) = e^{-\lambda_2 h} \eta(x)$, and hence

$$\|e^{\lambda_2 n} \mathbb{P}_x(X_{n+h} \in \cdot, X_{n+h} \neq \partial) - e^{-\lambda_2 h} \eta(x) \nu_2(\cdot \cap D_2)\|_{TV} \xrightarrow{n \rightarrow +\infty} 0$$

Since the convergence holds uniformly in $h \in [0, 1]$, this concludes the proof of (4) in the continuous time setting. The uniqueness of the quasi-stationary distribution is immediate, since any quasi-stationary distribution for $(X_t)_{t \in \mathbb{R}_+}$ is also a quasi-stationary distribution for $(X_n)_{n \in \mathbb{Z}_+}$. \square

Let us do some remarks before passing to the second set of assumptions.

Remark 2.1. In the proof, the speed of convergence in the above theorem are explicit in terms of the quantities appearing in the assumptions. In particular, if Assumption QSD2 holds true with

$$e^{(\lambda_2 + \varepsilon)t} \sup_{x \in D_1} \mathbb{P}_x(X_t \in D_1) \xrightarrow{t \rightarrow +\infty} 0,$$

for some $\varepsilon > 0$, then the convergence in (4) is exponential.

Remark 2.2. Non-uniform speed of convergence can also be proved under weaker form of Assumption QSD2. For instance if (3) holds true and if, for some $x \in D_1$, $e^{\lambda_2 t} \mathbb{P}_x(X_t \in D_1) \leq f(t)$ with f non-increasing and L^1 on \mathbb{R}_+ (but non-uniformly in $x \in D_1$), then,

$$\|e^{\lambda_2 n} \mathbb{P}_x(X_n \in \cdot) - \eta(x) \nu_2(\cdot \cap D_2)\|_{TV} \xrightarrow{n \rightarrow +\infty} 0.$$

Remark 2.3. The use of the absorbed Markov process setting is for convenience only: the above result applies more generally to semi-groups on $L_\infty(D)$ admitting an isolated simple leading eigenvalue $\lambda \in \mathbb{C}$.

2.2 Exponential convergence on D_1 and faster absorption in D_2

Let us now state our second set of assumptions.

Assumption QSD1. There exist a positive function η_1 on D_1 , a probability measure ν_1 on D_1 , a constant $\lambda_1 > 0$, and positive constants $C_1, \gamma_1 > 0$ such that

$$\|e^{\lambda_1 t} \mathbb{P}_x(X_t \in \cdot, X_t \in D_1) - \eta_1(x) \nu_1(\cdot)\|_{TV} \leq C_1 e^{-\gamma_1 t}, \quad \forall x \in D_1, t \geq 0. \quad (9)$$

In addition, for all $t \geq 0$,

$$e^{\lambda_1 t} \sup_{x \in D_2} \mathbb{P}_x(X_t \in D_2) \leq f(t) \quad (10)$$

where f is non-decreasing and L^1 on \mathbb{R}_+ .

Assumption QSD1 is very similar to Assumption QSD2, except that this assumption now deals with the quasi-stationarity for the restriction of the process X considered as absorbed by $D_2 \cup \{\partial\}$. Similarly to Assumption QSD2, Assumption QSD1 entails that η_1 is bounded, $\eta_1(x) = \lim_{t \rightarrow \infty} e^{\lambda_1 t} \mathbb{P}_x(X_t \in D_1)$ for all $x \in D_1$, and that ν_1 is a quasi-stationary distribution for the process X considered as absorbed by $D_2 \cup \{\partial\}$. The condition (10) tells that $\sup_{x \in D_2} \mathbb{P}_x(X_t \neq \partial)$ vanishes faster than the probability of survival starting from the quasi-stationary distribution ν_1 .

Theorem 2.2. *Under Assumption QSD1, there exists a positive finite measure ν on D such that*

$$\sup_{x \in D} \|e^{\lambda_1 t} \mathbb{P}_x(X_t \in \cdot, X_t \neq \partial) - \eta(x) \nu\|_{TV} \xrightarrow{t \rightarrow +\infty} 0, \quad (11)$$

where $\eta(x) = \eta_1(x)$ for all $x \in D_1$ and $\eta(x) = 0$ for all $x \in D_2$. In addition, $\nu/\nu(D)$ is the unique quasi-stationary distribution of X such that $\nu(D_1)/\nu(D) > 0$.

Hence, (11) entails that $\mathbb{P}_x(X_t \in \cdot | X_t \neq \partial)$ converges in total variation towards $\nu/\nu(D)$. Note also that Assumption QSD1 does not tell anything on the convergence of $\mathbb{P}_x(X_t \in \cdot | X_t \neq \partial)$ when $x \in D_2$.

Remark 2.4. Similarly as in Remark 2.1, if

$$e^{(\lambda_1 + \epsilon)t} \sup_{x \in D_2} \mathbb{P}_x(X_k \in D_2) \rightarrow 0$$

for some $\epsilon > 0$, then the convergence in (11) is exponential.

Remark 2.5. Similarly as in Remark 2.2, if (9) holds true and if

$$\sum_{k=0}^{\infty} e^{\lambda_1 k} \sup_{x \in D_2} \mathbb{P}_x(X_k \in A) < +\infty.$$

for some measurable $A \subset D$, then

$$\sup_{x \in D_1} |e^{\lambda_1 n} \mathbb{P}_x(X_n \in A) - \eta_1(x) \nu(A)| \xrightarrow{n \rightarrow +\infty} 0.$$

This last case is particularly interesting, since it applies to situations where ν is not necessarily a finite measure (one may have $\nu(D_2) = +\infty$).

Proof of Theorem 2.2. We only prove the result in the discrete-time setting. The adaptation to the continuous-time setting follows from the same argument as in the proof of Theorem 2.1.

Fix $x \in D_1$. The result is an immediate consequence of (9) if $A \subset D_1$ with $\nu(\cdot \cap D_1) = \nu_1$. It remains to consider the case $A \subset D_2$, the general case being obtained by linearity. For all measurable $A \subset D_2$ and all $x \in D_1$, we have, for all $k \in \{0, \dots, n-1\}$,

$$\begin{aligned} & \left| \mathbb{E}_x \left(\mathbf{1}_{\tau_1^c = n-k} \mathbb{P}_{X_{n-k}}(X_k \in A) \right) - \eta_1(x) e^{-\lambda_1(n-k-1)} \mathbb{E}_{\nu_1} \left(\mathbf{1}_{\tau_1^c = 1} \mathbf{1}_{X_{k+1} \in A} \right) \right| \\ &= \left| \mathbb{E}_x \left(f_A(X_{n-k-1}) \mathbf{1}_{n-k-1 < \tau_1^c} \right) - \eta_1(x) e^{-\lambda_1(n-k-1)} \nu_1(f_A) \right| \end{aligned}$$

where, for all $y \in D_1$, $f_A(y) = \mathbb{P}_y(X_1 \notin D_1, X_{k+1} \in A)$, and where τ_1^c was defined in the proof of Theorem 2.1. By Markov's property, we have $f_A(y) \leq \sup_{z \in D_2} \mathbb{P}_z(X_k \in A)$. Hence, according to (9), we have

$$\begin{aligned} & \left| \mathbb{E}_x \left(\mathbf{1}_{\tau_1^c = n-k} \mathbb{P}_{X_{n-k}}(X_k \in A) \right) - \eta_1(x) e^{-\lambda_1(n-k-1)} \mathbb{E}_{\nu_1} \left(\mathbf{1}_{\tau_1^c = 1} \mathbf{1}_{X_{k+1} \in A} \right) \right| \\ & \leq C_1 e^{-\lambda_1 n} e^{-\gamma_1(n-k-1)} \left(e^{\lambda_1(k+1)} \sup_{z \in D_2} \mathbb{P}_z(X_k \in A) \right) \\ & \leq C_1 e^{-\lambda_1 n} e^{-\gamma_1(n-k-1)} e^{\lambda_1 k} f(k). \end{aligned}$$

Summing over $k \in \{0, \dots, n-1\}$ and multiplying by $e^{\lambda_1 n}$, we get

$$\begin{aligned} & \left| e^{\lambda_1 n} \mathbb{P}_x(X_n \in A) - \eta_1(x) \sum_{k=0}^{n-1} e^{\lambda_1(k+1)} \mathbb{E}_{\nu_1} \left(\mathbf{1}_{\tau_1^c = 1} \mathbf{1}_{X_{k+1} \in A} \right) \right| \\ & \leq C_1 e^{-\gamma_1 n} \sum_{k=0}^{n-1} e^{\gamma_1(k+1)} e^{\lambda_1(k+1)} \sup_{z \in D_2} \mathbb{P}_z(X_k \in A) \\ & \leq C_1 e^{\lambda_1} \sum_{k=0}^{n-1} e^{-\gamma_1(n-k-1)} f(k), \end{aligned}$$

where the right hand term goes to 0 when $n \rightarrow +\infty$. Finally, we observe that

$$\begin{aligned} \sum_{k=n}^{+\infty} e^{\lambda_1(k+1)} \mathbb{E}_{\nu_1} (1_{\tau_1^c=1} 1_{X_{k+1} \in A}) &\leq \sum_{k=n}^{+\infty} e^{\lambda_1(k+1)} \sup_{z \in D_2} \mathbb{P}_x(X_k \in A) \\ &\leq e^{\lambda_1} \int_{n-1}^{+\infty} f(t) dt, \end{aligned}$$

which also goes to 0 when $n \rightarrow +\infty$ under the assumption (10). This concludes the proof of (11) with

$$\nu(A) := \nu_2(A \cap D_1) + \sum_{k=0}^{+\infty} e^{\lambda_1(k+1)} \mathbb{E}_{\nu_1} (1_{\tau_1^c=1} 1_{X_{k+1} \in A \cap D_2})$$

In particular, $\nu/\nu(D)$ is a quasi-limiting distribution of X and is thus a quasi-stationary distribution.

To conclude, let ν' be a quasi-stationary distribution for X such that $\nu'(D_1) > 0$. Integrating (11) with respect to ν' and noting that $\nu'(\eta_1) > 0$, we deduce that the exponential absorption rate of ν' is λ_1 and that $\nu/\nu(D)$ is a quasi-limiting distribution for X starting from ν' , and thus $\nu = \nu'$. \square

2.3 Exponential convergence in D_1 and D_2 with the same rate

Let us now present our last set of assumptions.

Assumption QSD1-2. There exist two positive functions η_1 on D_1 and η_2 on D_2 , two probability measures ν_1 on D_1 and ν_2 on D_2 , a positive constant $\lambda_0 > 0$, and positive constants $C_1, \gamma_1, C_2, \gamma_2 > 0$ such that (9) and (3) hold true with $\lambda_1 = \lambda_2 = \lambda_0$.

In other terms, ν_2 and ν_1 are the quasi-stationary distributions for X respectively started from D_2 and absorbed at ∂ , and started from D_1 and absorbed at $D_2 \cup \{\partial\}$, associated to the same absorption rate $\lambda_0 > 0$. Under this assumption, we have the following result.

Theorem 2.3. *Under Assumption QSD1-2, the process admits $\nu_2(\cdot \cap D_2)$ as unique quasi-stationary distribution and*

$$\sup_{x \in D_1} \left\| \frac{e^{\lambda_0 t}}{t} \mathbb{P}_x(X_t \in \cdot, X_t \in D_2) - \eta(x) \nu_2(\cdot \cap D_2) \right\|_{TV} \leq \frac{C}{t+1}, \quad (12)$$

where η is a positive function on D_1 and C is a positive constant.

In particular, Malthusian behavior (3) does not hold for $x \in D_1$. However, (12) still entails that the probability measure $\mathbb{P}_x(X_t \in \cdot | X_t \neq \partial)$ converges in total variation towards ν_2 for all $x \in D_1$ (and also for all $x \in D_2$ by Hypothesis (3)).

Proof of Theorem 2.3. As for the proof of Theorem 2.1, we only deal with the discrete-time setting.

Fix $x \in D_1$ and measurable set $A \subset D_2$. For all $k \geq 0$, we have, using (3) (recall that $\lambda_1 = \lambda_2 = \lambda_0$),

$$\left| e^{\lambda_0(k+1)} \mathbb{P}_{\nu_1}(\tau_1^c = 1, X_{k+1} \in A) - e^{\lambda_0} \mathbb{E}_{\nu_1}(1_{\tau_1^c=1} \eta_2(X_1)) \nu_2(A) \right| \leq e^{\lambda_0} C_2 e^{-\gamma_2 k}, \quad (13)$$

where τ_1^c was defined in the proof of Theorem 2.1. Moreover, for all $n \geq 1$ and $k \in \{0, \dots, n-1\}$,

$$\begin{aligned} & \left| e^{\lambda_0(k+1)} \eta_1(x) \mathbb{P}_{\nu_1}(\tau_1^c = 1, X_{k+1} \in A) - e^{\lambda_0 n} \mathbb{E}_x(1_{\tau_1^c=n-k} 1_{X_n \in A}) \right| \\ &= e^{\lambda_0(k+1)} \left| \eta_1(x) \nu_1(f_A) - e^{\lambda_0(n-k-1)} \mathbb{E}_x(f_A(X_{n-k-1})) \right|, \end{aligned}$$

where $f_A(y) = \mathbb{P}_y(\tau_1^c = 1, X_{k+1} \in A) \leq \sup_{z \in D_2} \mathbb{P}_z(X_k \in D_2)$. Hence using (9), we deduce that

$$\begin{aligned} & \left| e^{\lambda_0(k+1)} \eta_1(x) \mathbb{P}_{\nu_1}(\tau_c^1 = 1, X_{k+1} \in A) - e^{\lambda_0 n} \mathbb{E}_x(1_{\tau_1^c=n-k} 1_{X_n \in A}) \right| \\ & \leq e^{\lambda_0(k+1)} C_1 e^{-\gamma_1(n-k-1)} \sup_{z \in D_2} \mathbb{P}_z(X_k \in D_2) \\ & = C_1 e^{\lambda_0} e^{-\gamma_1(n-k-1)} \sup_{z \in D_2} e^{\lambda_0 k} \mathbb{P}_z(X_k \in D_2), \end{aligned}$$

where, according to (3), $\sup_{z \in D_2} e^{\lambda_0 k} \mathbb{P}_z(X_k \in D_2)$ is uniformly bounded in k by $C_2 + \|\eta_2\|_\infty$. This, (13) and summing over $k \in \{0, \dots, n-1\}$ imply that

$$\begin{aligned} & \left| e^{\lambda_0 n} \mathbb{P}_x(X_n \in A) - n e^{\lambda_0} \eta_1(x) \mathbb{E}_{\nu_1}(1_{\tau_1^c=1} \eta_2(X_1)) \nu_2(A) \right| \\ & \leq C_2 \eta_1(x) e^{\lambda_0} \sum_{k=0}^{\infty} e^{-\gamma_2 k} + \sum_{k=0}^{+\infty} e^{-\gamma_1 k} C_1 e^{\lambda_0} (C_2 + \|\eta_2\|_\infty). \end{aligned}$$

This concludes the proof of (12) with $\eta(x) := e^{\lambda_0} \eta_1(x) \mathbb{E}_{\nu_1}(1_{\tau_1^c=1} \eta_2(X_1))$. \square

3 Proof of Theorem 1.1

We start with a proposition related to the theory of quasi-stationary distributions for diffusion processes.

Proposition 3.1. *There exist a positive function $\eta_2 : [3, 5) \rightarrow (0, +\infty)$, a probability measure ν_2 on $[3, 5)$ and positive constants $C_2, \lambda_2, \gamma_2 > 0$ such that, for all $\alpha > 0$ and for all probability measure μ on $[3, 5)$,*

$$\|e^{\alpha\lambda_2 t} \mathbb{P}_\mu(X_t^\alpha \in \cdot \cap [3, 5)) - \mu(\eta_2)\nu_2(\cdot)\|_{TV} \leq C_2 e^{-\alpha\gamma_2 t}, \quad \forall t \geq 0. \quad (14)$$

There exist a positive function $\eta_1 : (0, 2) \rightarrow (0, +\infty)$, a probability measure ν_1 on $(0, 2)$ and positive constants $C_1, \lambda_1, \gamma_1 > 0$ such that, for all probability measure μ on $(0, 2)$ and all $\alpha > 0$,

$$\|e^{\lambda_1 t} \mathbb{P}_\mu(X_t^\alpha \in \cdot \cap (0, 2)) - \mu(\eta_1)\nu_1(\cdot)\|_{TV} \leq C_1 e^{-\gamma_1 t}, \quad \forall t \geq 0. \quad (15)$$

Proof of Proposition 3.1. For any $\alpha > 0$, on the event $X_0^\alpha \in [3, 5)$, the process remains almost surely in $[3, 5)$ until it reaches ∂D at the end point 5. It is known (this can be proved for instance using Section 4.5 of [8]) that, considering the process X^α restricted to $[3, 5)$ absorbed when it reaches 5, there exists a probability measure μ_α on $[3, 5)$ (the quasi-stationary distribution of X^α restricted to $[3, 5)$), a positive function $\zeta_\alpha : [3, 5) \rightarrow (0, +\infty)$, and positive constants $c_\alpha, \delta_\alpha, \delta'_\alpha > 0$ such that, for all probability measure μ on $[3, 5)$,

$$\|e^{\delta_\alpha t} \mathbb{P}_\mu(X_t^\alpha \in \cdot) - \mu(\zeta_\alpha)\mu_\alpha(\cdot)\|_{TV} \leq c_\alpha e^{-\delta'_\alpha t}, \quad \forall t \geq 0. \quad (16)$$

Also, since $Law((X_t^\alpha)_{t \geq 0}) = Law((X_{\alpha t}^1)_{t \geq 0})$, we deduce that $\zeta_\alpha = \zeta_1$, $\mu_\alpha = \mu_1$ and $\delta_\alpha = \alpha\delta_1$ for all $\alpha > 0$. Moreover, one can take $c_\alpha = c_1$ and $\delta'_\alpha = \alpha\delta'_1$. Setting $\eta_2 = \zeta_2$, $\nu_2 = \mu_1$, $C_2 = c_1$, $\lambda_2 = \delta_1$ and $\gamma_2 = \delta'_1$, this proves (14).

Similarly, the law of the process X^α restricted to $(0, 2)$ and absorbed when it reaches $\{0, 2\}$ does not depend on α , and there exists a probability measure ν_1 on $(0, 2)$ (the quasi-stationary distribution of the process X^α conditioned to remain in $(0, 2)$), a positive function $\eta_1 : (0, 2) \rightarrow (0, +\infty)$ and positive constants $C_1, \lambda_1, \gamma_1 > 0$ such that, for all probability measure μ on $(0, 2)$ and all $\alpha > 0$,

$$\|e^{\lambda_1 t} \mathbb{P}_\mu(X_t^\alpha \in \cdot, t < T_0 \wedge T_2) - \mu(\eta_1)\nu_1(\cdot)\|_{TV} \leq C_1 e^{-\gamma_1 t}, \quad \forall t \geq 0, \quad (17)$$

where T_a denotes the first hitting time of $\{a\}$ (see again Section 4.5 in [8]). Since the process cannot enter $(0, 2)$ after time $T_0 \wedge T_2$, we deduce that $\mathbb{P}_\mu(X_t^\alpha \in \cdot, t < T_0 \wedge T_2) = \mathbb{P}_\mu(X_t^\alpha \in \cdot \cap (0, 2))$. This concludes the proof of Proposition 3.1. \square

We consider now the behaviour of the process with initial position in $[2, 3)$. For any $\alpha > 0$ and $x \in [2, 3)$, denote by $(f_t^\alpha(x))_{t \geq 0}$ the solution of the ODE $\partial f_t^\alpha(x)/\partial t = \psi^1(f_t^\alpha(x)) + \alpha\psi^2(f_t^\alpha(x))$ and $f_0^\alpha(x) = x$. For all $x \in [2, 3)$, denote by $t_3(x) = \inf\{t \geq 0, f_t^\alpha(x) = 3\}$, then

$$\mathbb{P}_x(X_t^\alpha \in A) = \begin{cases} \mathbf{1}_{f_t^\alpha(x) \in A} & \text{if } f_t^\alpha(x) < 3, \\ \mathbb{P}_3(X_{t-t_3(x)}^\alpha \in A) & \text{if } f_t^\alpha(x) \geq 3, \end{cases}$$

where we used the strong Markov property at time T_3 and the fact that X^α is deterministic with drift equal to $\psi^1 + \alpha\psi^2$ on $[2, 3]$ (so that $T_3 = t_3(x)$ \mathbb{P}_x -almost surely). In particular, we deduce from Proposition 3.1 that, for all $x \in [2, 3)$ and $t \geq t_3(x)$,

$$\|e^{\alpha\lambda_2(t-t_3(x))}\mathbb{P}_x(X_t^\alpha \in \cdot) - \eta_2(3)\nu_2(\cdot)\|_{TV} \leq C_2 e^{-\alpha\gamma_2(t-t_3(x))}$$

and hence

$$\|e^{\alpha\lambda_2 t}\mathbb{P}_x(X_t^\alpha \in \cdot) - \eta_\alpha(x)\nu_2(\cdot)\|_{TV} \leq e^{\alpha(\lambda_2+\gamma_2)t_3(x)}C_2 e^{-\alpha\gamma_2 t} \leq C'_2 e^{-\alpha\gamma_2 t},$$

where $\eta_\alpha(x) := e^{\alpha\lambda_2 t_3(x)}\eta_2(3)$ and $C'_2 = e^{\alpha(\lambda_2+\gamma_2)t_3(2)}C_2$. By integration of the last inequality and by (14), we thus proved that, for any initial distribution in $[2, 5)$,

$$\|e^{\alpha\lambda_2 t}\mathbb{P}_\mu(X_t^\alpha \in \cdot) - \mu(\eta_\alpha)\nu_2(\cdot)\|_{TV} \leq C'_2 e^{-\alpha\gamma_2 t}, \quad \forall t \geq 0, \quad (18)$$

where

$$\eta_\alpha(x) = \begin{cases} e^{\alpha\lambda_2 t_3(x)}\eta_2(3) & \text{if } x \in [2, 3] \\ \eta_2(x) & \text{if } x \in [3, 5). \end{cases}$$

We are now in a position to apply the results of Section 2, with $D_1 = (0, 2)$ and $D_2 = [2, 5)$.

Case $\alpha < \lambda_1/\lambda_2$. In this case we observe that Assumption QSD2 is satisfied (of course with $\alpha\lambda_2$ instead of λ_2). Indeed, on the one hand (3) is immediate from (18), while, for all $x \in D_1$,

$$\sup_{x \in D_1} e^{\alpha\lambda_2 t} \mathbb{P}_x(X_t \in D_1) \leq e^{(\alpha\lambda_2 - \lambda_1)t} (C_1 + \|\eta_1\|_\infty) \xrightarrow{t \rightarrow +\infty} 0$$

and is $L^1(\mathbb{R}_+)$, where we used (15). Moreover, we have $\mathbb{P}_x(X_1 \in D_2) > 0$ for all $x \in D_1$ and hence, according to Theorem 2.1, $\nu_2(\cdot \cap D_2)$ is the unique quasi-stationary distribution ν for the process X^α absorbed at $\{0, 5\}$, and (4) implies that, for all probability measure μ on D ,

$$\Phi_t^\alpha(\mu) \xrightarrow[t \rightarrow +\infty]{TV} \nu_2(\cdot \cap D_2).$$

Case $\alpha = \lambda_1/\lambda_2$. In this case, we observe that Assumption QSD1-2 is satisfied. To deduce the convergence of $\Phi_t^\alpha(\mu)$, we need to distinguish two cases: either $\mu(D_1) = 0$ and then the fact that

$$\Phi_t^\alpha(\mu) \xrightarrow[t \rightarrow +\infty]{TV} \nu_2(\cdot \cap D_2).$$

follows from (3), or $\mu(D_1) > 0$ and then it follows from (9), (3) and (12) that, on the one hand,

$$\frac{e^{\lambda_0 t}}{t} \mathbb{P}_\mu(X_t \neq \partial) = \frac{e^{\lambda_0 t}}{t} \mathbb{P}_\mu(X_t \in D_1) + \frac{e^{\lambda_0 t}}{t} \mathbb{P}_\mu(X_t \in D_2, X_t \neq \partial) \xrightarrow{t \rightarrow +\infty} \mu(\eta) > 0$$

and on the other hand, that for all $A \subset D_1 \cup D_2$ measurable,

$$\frac{e^{\lambda_0 t}}{t} \mathbb{P}_\mu(X_t \in A) = \frac{e^{\lambda_0 t}}{t} \mathbb{P}_\mu(X_t \in A \cap D_1) + \frac{e^{\lambda_0 t}}{t} \mathbb{P}_\mu(X_t \in A \cap D_2) \xrightarrow{t \rightarrow +\infty} \mu(\eta) \nu_2(A \cap D_2),$$

where the convergence is uniform with respect to A . Hence, in all cases,

$$\Phi_t^\alpha(\mu) \xrightarrow[t \rightarrow +\infty]{TV} \nu_2(\cdot \cap D_2).$$

Case $\alpha > \lambda_1/\lambda_2$. In this case, we observe that Assumption QSD1 is satisfied. Indeed, on the one hand (9) holds true, while, for all $x \in D_2$,

$$\sum_{k \in \mathbb{Z}_+} \sup_{x \in D_2} e^{\lambda_1 k} \mathbb{P}_x(X_k \in D_2) \leq \sum_{k \in \mathbb{Z}_+} e^{(\lambda_1 - \alpha\lambda_2)k} (C'_2 + \|\eta_\alpha\|_\infty) < +\infty,$$

where we used (18) and $\alpha\lambda_2 > \lambda_1$. Then, (11) in Theorem 2.2 implies that there exists a probability measure ν on D such that $\nu(D_1) > 0$ and such that, for all probability measure μ on D such that $\mu(D_1) > 0$,

$$\Phi_t^\alpha(\mu) \xrightarrow[t \rightarrow +\infty]{TV} \nu.$$

In addition, (18) entails that, for all probability measure μ on D such that $\mu(D_1) = 0$,

$$\Phi_t^\alpha(\mu) \xrightarrow[t \rightarrow +\infty]{TV} \nu_2(\cdot \cap D_2).$$

This concludes the proof of Theorem 1.1.

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